18

Solution of Triangles

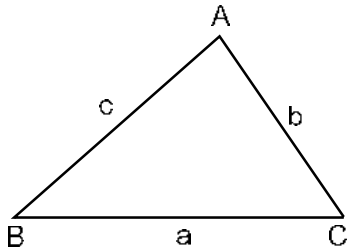
INTRODUCTION The three sides and three angles of a triangle are called the *parts of the triangle.*

When at least three parts of a triangle are known (i.e., *any* three parts), we can compute the measures of the remaining three parts.

The process of computing the measures of unknown parts of a triangle from those of the given parts is known as solving the triangle.

THE OBLIQUE TRIANGLE *A triangle which does not contain a right angle is called an oblique triangle.*

In a $\triangle ABC$, we denote the lengths of the sides as $BC = a$, $CA = b$ and $AB = c$.



THEOREM 1 (Sine Formula or Sine Rule) *In any triangle, the sides are proportional to the sines of the opposite angles.*
That is, in a $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

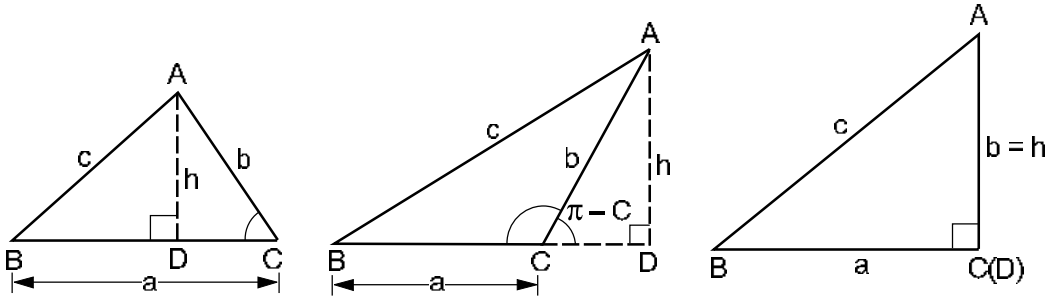
PROOF Let $\triangle ABC$ be any triangle.

We know that the sum of all the angles of a triangle is 180°. So, each angle of a triangle cannot be obtuse.

Let us assume here that $\angle B$ is acute.

Now, we consider the following cases:

(i) $\angle C$ is acute (ii) $\angle C$ is obtuse (iii) $\angle C$ is a right angle



(i) (ii) (iii)

Draw $AD \perp BC$ or BC produced.

Let $AD = h$.

Then, in each figure, we have:

$$\frac{AD}{AB} = \sin B \Rightarrow \frac{h}{c} = \sin B \Rightarrow h = c\sin B \qquad \ldots (1)$$

Also, we have

In Fig. (i), $\frac{AD}{AC} = \sin C \Rightarrow \frac{h}{b} = \sin C \Rightarrow h = b\sin C.$

In Fig. (ii), $\frac{AD}{AC} = \sin(\pi - C) = \sin C \Rightarrow \frac{h}{b} = \sin C \Rightarrow h = b\sin C.$

In Fig. (iii), $\frac{AD}{AC} = 1 = \sin\frac{\pi}{2} = \sin C \Rightarrow \frac{h}{b} = \sin C \Rightarrow b\sin C.$

Thus, in each case, we have: $h = b\sin C$ $\qquad \ldots (2)$

From (1) and (2), we get:

$$c\sin B = b\sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, $\frac{b}{\sin B} = \frac{a}{\sin A}$.

Hence, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

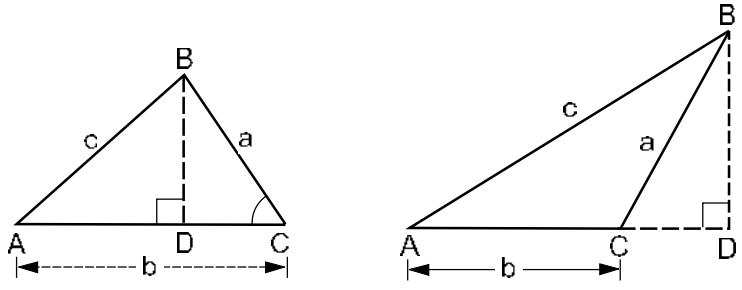
THEOREM 2 (Cosine Formulae) *Let a, b and c be the lengths of sides of $\triangle ABC$, opposite to $\angle A$, $\angle B$ and $\angle C$ respectively. Then,*

(i) $a^2 = b^2 + c^2 - 2bc\cos A$

(ii) $b^2 = c^2 + a^2 - 2ca\cos B$

(iii) $c^2 = a^2 + b^2 - 2ab\cos C$

PROOF Let $\triangle ABC$ be given, as shown in Fig. (i) and Fig. (ii).



(i) (ii)

From Fig. (i), we get

$$\begin{aligned} BC^2 &= BD^2 + DC^2 \\ &= BD^2 + (AC - AD)^2 \\ &= BD^2 + AD^2 + AC^2 - 2AC \cdot AD \\ &= AB^2 + AC^2 - 2AC \cdot AB\cos A \\ &\qquad [\because BD^2 + AD^2 = AB^2 \text{ and } AD = AB\cos A] \\ &= AC^2 + AB^2 - 2AC \cdot AB\cos A \end{aligned}$$

$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A.$

From Fig. (ii), we get

$$BC^2 = BD^2 + CD^2$$
$$= BD^2 + (AD - AC)^2$$
$$= BD^2 + AD^2 + AC^2 - 2AC \cdot AD$$
$$= AB^2 + AC^2 - 2AC \cdot AB\cos A$$
$$[\because\ BD^2 + AD^2 = AB^2 \text{ and } AD = AB\cos A]$$

$\Rightarrow\ a^2 = b^2 + c^2 - 2bc\cos A.$

Hence, from both the cases, we get

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Similarly, $b^2 = c^2 + a^2 - 2ca\cos B$ and $c^2 = a^2 + b^2 - 2ab\cos C.$

REMARKS *A convenient method writing cosine formulae is given below:*

$$(i)\ \cos A = \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$(ii)\ \cos B = \frac{(c^2 + a^2 - b^2)}{2ca}$$

$$(iii)\ \cos C = \frac{(a^2 + b^2 - c^2)}{2ab}$$

SUMMARY

1. **Sine Formula**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

2. **Cosine Formulae**

(i) $a^2 = b^2 + c^2 - 2bc\cos A$

(ii) $b^2 = c^2 + a^2 - 2ca\cos B$

(iii) $c^2 = a^2 + b^2 - 2ab\cos C$

THEOREM 3 **(Napier's Analogies)** *In any $\triangle ABC$, prove that*

$$(i)\ \tan\frac{(B-C)}{2} = \frac{(b-c)}{(b+c)}\cot\frac{A}{2}$$

$$(ii)\ \tan\frac{(C-A)}{2} = \frac{(c-a)}{(c+a)}\cot\frac{B}{2}$$

$$(iii)\ \tan\frac{(A-B)}{2} = \frac{(a-b)}{(a+b)}\cot\frac{C}{2}$$

PROOF By the sine formula, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}.$$

Then, $a = k\sin A$, $b = k\sin B$ and $c = k\sin C.$

(i) $\therefore\ \dfrac{(b-c)}{(b+c)} = \dfrac{k\sin B - k\sin C}{k\sin B + k\sin C}$

$$= \frac{(\sin B - \sin C)}{(\sin B + \sin C)} = \frac{2\cos\frac{(B+C)}{2}\sin\frac{(B-C)}{2}}{2\sin\frac{(B+C)}{2}\cos\frac{(B-C)}{2}}$$

$$= \cot\frac{(B+C)}{2}\tan\frac{(B-C)}{2}$$

$$= \cot\left(\frac{\pi}{2} - \frac{A}{2}\right)\tan\frac{(B-C)}{2} \qquad \left[\because \frac{A}{2} + \frac{(B+C)}{2} = \frac{\pi}{2}\right]$$

$$= \tan\frac{A}{2}\tan\frac{(B-C)}{2}$$

$$\Rightarrow \quad \tan\frac{(B-C)}{2} = \frac{(b-c)}{(b+c)}\cot\frac{A}{2}.$$

Similarly, (ii) and (iii) may be proved.

SOLVED EXAMPLES

EXAMPLE 1 *In any $\triangle ABC$, prove that*

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$$\Rightarrow \quad a = k\sin A,\ b = k\sin B \text{ and } c = k\sin C.$$

$$\therefore \quad \text{LHS} = a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B)$$

$$= k\sin A(\sin B - \sin C) + k\sin B(\sin C - \sin A) + k\sin C(\sin A - \sin B)$$

$$= k[(\sin A \sin B - \sin A \sin C) + (\sin B \sin C - \sin A \sin B) + (\sin A \sin C - \sin B \sin C)]$$

$$= (k \times 0) = 0 = \text{RHS}.$$

EXAMPLE 2 *In any $\triangle ABC$, prove that*

$$a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

$$\Rightarrow \quad \sin A = ka,\ \sin B = kb \text{ and } \sin C = kc.$$

Substituting the values of $\sin B$ and $\sin C$ and using cosine formulae, we have

$$a\sin(B-C) = a(\sin B\cos C - \cos B\sin C)$$

$$= a\left[kb\cdot\frac{(a^2+b^2-c^2)}{2ab} - kc\cdot\frac{(c^2+a^2-b^2)}{2ac}\right]$$

$$= \frac{k}{2}\cdot[(a^2+b^2-c^2)-(c^2+a^2-b^2)] = k(b^2-c^2)].$$

Similarly, $b\sin(C-A) = b(\sin C\cos A - \cos C\sin A) = k(c^2-a^2)$.

And, $c\sin(A-B) = c(\sin A\cos B - \cos A\sin B) = k(a^2-b^2)$.

$\therefore$ $a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$

$= k(b^2-c^2) + k(c^2-a^2) + k(a^2-b^2)$

$= k(b^2-c^2+c^2-a^2+a^2-b^2) = k\times 0 = 0.$

EXAMPLE 3 *In any $\triangle ABC$, prove that*

$$\frac{\sin(B-C)}{\sin(B+C)} = \frac{(b^2-c^2)}{a^2}.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\,(\text{say})$$

$\Rightarrow$ $a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

$\therefore$ $\text{RHS} = \dfrac{(b^2-c^2)}{a^2} = \dfrac{k^2\sin^2 B - k^2\sin^2 C}{k^2\sin^2 A}$

$$= \frac{(\sin^2 B - \sin^2 C)}{\sin^2 A} = \frac{\sin(B+C)\cdot\sin(B-C)}{\sin^2(B+C)}$$

$$\left[\because (A+B+C) = \pi \Rightarrow A = \pi-(B+C)\right.$$
$$\left.\therefore \sin A = \sin\{\pi-(B+C)\} = \sin(B+C)\right]$$

$$= \frac{\sin(B-C)}{\sin(B+C)} = \text{LHS}.$$

$\therefore$ $\text{RHS} = \text{LHS}.$

Hence, $\dfrac{\sin(B-C)}{\sin(B+C)} = \dfrac{(b^2-c^2)}{a^2}$.

EXAMPLE 4 *In any $\triangle ABC$, prove that*

$$\frac{(a-b)}{c}\cos\frac{C}{2} = \sin\frac{(A-B)}{2}.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\,(\text{say})$$

$\Rightarrow$ $a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

$$\therefore \quad \text{LHS} = \frac{(a-b)}{c}\cos\frac{C}{2}$$

$$= \frac{(k\sin A - k\sin B)}{k\sin C}\cos\frac{C}{2} = \frac{k(\sin A - \sin B)}{k\sin C}\cdot\cos\frac{C}{2}$$

$$= \frac{(\sin A - \sin B)}{\sin C}\cdot\cos\frac{C}{2} = \frac{2\cos\dfrac{(A+B)}{2}\sin\dfrac{(A-B)}{2}}{2\sin\dfrac{C}{2}\cos\dfrac{C}{2}}\cdot\cos\frac{C}{2}$$

$$\left[\because\ (\sin C - \sin D) = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}\right]$$

$$= \frac{\cos\left(\dfrac{\pi}{2}-\dfrac{C}{2}\right)\cdot\sin\dfrac{(A-B)}{2}}{\sin\dfrac{C}{2}} \qquad \left[\because\ \frac{A+B}{2} = \left(\frac{\pi}{2}-\frac{C}{2}\right)\right]$$

$$= \frac{\sin\dfrac{C}{2}\cdot\sin\dfrac{(A-B)}{2}}{\sin\dfrac{C}{2}} = \sin\frac{(A-B)}{2} = \text{RHS}.$$

Hence, $\dfrac{(a-b)}{c}\cos\dfrac{C}{2} = \sin\dfrac{(A-B)}{2}$.

EXAMPLE 5 *In any $\triangle ABC$, prove that*

$$\frac{(b^2-c^2)}{a^2}\sin 2A + \frac{(c^2-a^2)}{b^2}\sin 2B + \frac{(a^2-b^2)}{c^2}\sin 2C = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow \quad a = k\sin A,\ b = k\sin B$ and $c = k\sin C$.

Applying sine rule and cosine formula, we get:

$$\frac{(b^2-c^2)}{a^2}\sin 2A = \frac{(b^2-c^2)}{a^2}\cdot(2\sin A\cos A)$$

$$= \frac{(b^2-c^2)}{a^2}\left(\frac{2a}{k}\right)\cdot\frac{(b^2+c^2-a^2)}{2bc}$$

$$\left[\because\ \sin A = \frac{a}{k} \text{ and } \cos A = \frac{(b^2+c^2-a^2)}{2bc}\right]$$

$$= \frac{1}{(kabc)}\cdot(b^2-c^2)(b^2+c^2-a^2) \qquad \ldots \text{(i)}$$

Similarly, $\dfrac{(c^2-a^2)}{b^2}\sin 2B = \dfrac{1}{(kabc)}\cdot(c^2-a^2)\,(c^2+a^2-b^2)$ $\qquad \ldots \text{(ii)}$

And, $\frac{(a^2-b^2)}{c^2}\sin 2C = \frac{1}{(kabc)}\cdot(a^2-b^2)\,(a^2+b^2-c^2)$. … (iii)

From (i), (ii) and (iii), we get

$$\text{LHS} = \frac{(b^2-c^2)}{a^2}\sin 2A + \frac{(c^2-a^2)}{b^2}\sin 2B + \frac{(a^2-b^2)}{c^2}\sin 2C$$

$$= \frac{1}{(kabc)}\cdot[(b^2-c^2)(b^2+c^2-a^2) + (c^2-a^2)(c^2+a^2-b^2) + (a^2-b^2)(a^2+b^2-c^2)]$$

$$= \frac{1}{(kabc)}\times 0 = 0 = \text{RHS}.$$

Hence, $\frac{(b^2-c^2)}{a^2}\sin 2A + \frac{(c^2-a^2)}{b^2}\sin 2B + \frac{(a^2-b^2)}{c^2}\sin 2C = 0.$

EXAMPLE 6 *In any $\triangle ABC$, prove that*

$$\frac{(b-c)}{(b+c)} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow$ $a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

$\therefore$ $$\text{LHS} = \frac{(b-c)}{(b+c)} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C} = \frac{k(\sin B - \sin C)}{k(\sin B + \sin C)}$$

$$= \frac{(\sin B - \sin C)}{(\sin B + \sin C)} = \frac{2\cos\frac{(B+C)}{2}\sin\frac{(B-C)}{2}}{2\sin\frac{(B+C)}{2}\cos\frac{(B-C)}{2}}$$

$$= \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} = \text{RHS}.$$

Hence, $\frac{(b-c)}{(b+c)} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$

EXAMPLE 7 *In any $\triangle ABC$, prove that*

$$a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}.$$

SOLUTION Applying cosine formulae, we have

$$\text{LHS} = a(\cos C - \cos B)$$

$$= a\left[\frac{(a^2+b^2-c^2)}{2ab} - \frac{(a^2+c^2-b^2)}{2ac}\right] = \frac{(a^2c+b^2c-c^3-a^2b-bc^2+b^3)}{2bc}$$

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$$= \frac{(b^3 - c^3) + (b^2c - bc^2) - (a^2b - a^2c)}{2bc} = \frac{(b^3 - c^3) + bc(b - c) - a^2(b - c)}{2bc}$$

$$= (b - c)\,\frac{[(b^2 + c^2 + bc) + bc - a^2]}{2bc} = (b - c) \cdot \left\{\frac{(b^2 + c^2 - a^2)}{2bc} + \frac{2bc}{2bc}\right\}$$

$$= (b - c)\left\{\frac{(b^2 + c^2 - a^2)}{2bc} + 1\right\} = (b - c)(1 + \cos A)$$

$$= 2(b - c)\cos^2\frac{A}{2} = \text{RHS.}$$

$\therefore \quad a(\cos C - \cos B) = 2(b - c)\cos^2\dfrac{A}{2}$.

EXAMPLE 8 *In any $\triangle ABC$, prove that*

$$a\cos A + b\cos B + c\cos C = 2a\sin B \sin C.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow \quad a = k \sin A$, $b = k \sin B$ and $c = k \sin C$.

$\therefore \quad$ LHS $= a\cos A + b\cos B + c\cos C$

$$= k\sin A \cos A + k\sin B \cos B + k\sin C \cos C$$

$$= \frac{1}{2}k(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{1}{2}k[2\sin(A + B)\cos(A - B) + 2\sin C \cos C]$$

$$= k[\sin(A + B)\cos(A - B) + \sin C \cos C]$$

$$= k[\sin(\pi - C)\cos(A - B) + \sin C \cos C]$$

$$= k\sin C[\cos(A - B) + \cos C]$$

$$= k\sin C[\cos(A - B) + \cos\{\pi - (A + B)\}]$$

$$= k\sin C[\cos(A - B) - \cos(A + B)]$$

$$= k\sin C \times 2\sin A \sin B$$

$$= 2(k\sin A)\sin B \sin C$$

$$= 2a\sin B \sin C = \text{RHS.}$$

Hence, $a\cos A + b\cos B + c\cos C = 2a\sin B \sin C$.

EXAMPLE 9 *In any $\triangle ABC$, prove that*

$$(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

$\Rightarrow \quad \sin A = ka$, $\sin B = kb$ and $\sin C = kc$.

$$\therefore \quad (b^2 - c^2)\cot A = \frac{(b^2 - c^2) \cdot \cos A}{\sin A} = \frac{(b^2 - c^2)}{ka} \cdot \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$= \frac{1}{(2kabc)} \cdot (b^2 - c^2)(b^2 + c^2 - a^2) = \frac{(b^4 - c^4 - a^2b^2 + a^2c^2)}{2kabc}.$$

Similarly, $(c^2 - a^2) \cot B = \dfrac{(c^4 - a^4 - b^2c^2 + a^2b^2)}{2kabc}$.

And, $(a^2 - b^2) \cot C = \dfrac{(a^4 - b^4 - a^2c^2 + b^2c^2)}{2kabc}$.

$\therefore$ LHS $= (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C$

$$= \frac{1}{2kabc} \cdot [(b^4 - c^4 - a^2b^2 + a^2c^2) + (c^4 - a^4 - b^2c^2 + a^2b^2) + (a^4 - b^4 - a^2c^2 + b^2c^2)]$$

$$= \frac{1}{2kabc} \times 0 = 0 = \text{RHS}.$$

Hence, $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$.

EXAMPLE 10 *In any $\triangle ABC$, prove that*

$$a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow$ $a = k \sin A$, $b = k \sin B$ and $c = k \sin C$.

$\therefore$ $a^3 \sin(B - C) = k^3 \sin^3 A \cdot \sin(B - C)$

$= k^3 \sin^2 A \cdot \sin A \cdot \sin(B - C)$

$= k^3 \sin^2 A[\sin\{\pi - (B + C)\} \sin(B - C)]$

$[\because A + (B + C) = \pi]$

$= k^3 \sin^2 A[\sin(B + C) \sin(B - C)]$

$= k^3 \sin^2 A \cdot (\sin^2 B - \sin^2 C)$.

Similarly, $b^3 \sin(C - A) = k^3 \sin^2 B(\sin^2 C - \sin^2 A)$.

And, $c^3 \sin(A - B) = k^3 \sin^2 C(\sin^2 A - \sin^2 B)$.

$\therefore$ $a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B)$

$= k^3 \sin^2 A(\sin^2 B - \sin^2 C) + k^3 \sin^2 B(\sin^2 C - \sin^2 A) + k^3 \sin^2 C(\sin^2 A - \sin^2 B)$

$= 0$.

EXAMPLE 11 *In any $\triangle ABC$, prove that*

$$(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2.$$

SOLUTION We have

$$\text{LHS} = (a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2}$$

$$= a^2\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right) + b^2\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right)$$
$$- 2ab\cos^2\frac{C}{2} + 2ab\sin^2\frac{C}{2}$$
$$= a^2 + b^2 - 2ab\left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right)$$
$$= a^2 + b^2 - 2ab\cos C = c^2 = \text{RHS.} \qquad \text{[by cosine formula]}$$

$\therefore \quad (a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2} = c^2.$

EXAMPLE 12 *In a $\triangle ABC$, prove that*

$$(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow \quad a = k\sin A,\ b = k\sin B$ and $c = k\sin C.$

$\therefore \quad (b-c)\cot\frac{A}{2} = k(\sin B - \sin C)\cot\frac{A}{2}$

$$= 2k\cos\frac{(B+C)}{2}\sin\frac{(B-C)}{2}\cdot\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= 2k\cos\left(\frac{\pi}{2} - \frac{A}{2}\right)\sin\frac{(B-C)}{2}\cdot\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= 2k\sin\frac{A}{2}\sin\frac{(B-C)}{2}\cdot\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= 2k\sin\frac{(B-C)}{2}\cdot\cos\frac{A}{2}$$

$$= 2k\sin\frac{(B-C)}{2}\cdot\cos\left\{\frac{\pi}{2} - \frac{(B+C)}{2}\right\}$$

$$= 2k\sin\frac{(B-C)}{2}\cdot\sin\frac{(B+C)}{2}$$

$$= k(\cos C - \cos B).$$

Similarly, we have

$$(c-a)\cot\frac{B}{2} = k(\cos A - \cos C) \text{ and } (a-b)\cot\frac{C}{2} = k(\cos B - \cos A).$$

$\therefore \quad (b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2}$

$$= k\cdot[(\cos C - \cos B) + (\cos A - \cos C) + (\cos B - \cos A)] = 0.$$

EXAMPLE 13 *In a* $\triangle ABC$, *if* $\cos A = \dfrac{\sin B}{2\sin C}$, *show that the triangle is isosceles.*

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

$$\Rightarrow \quad \sin A = ka,\ \sin B = kb \text{ and } \sin C = kc.$$

$$\therefore \quad \cos A = \frac{\sin B}{2\sin C} \Rightarrow \frac{(b^2 + c^2 - a^2)}{2bc} = \frac{kb}{2kc}$$

$$\Rightarrow (b^2 + c^2 - a^2) = b^2$$

$$\Rightarrow c^2 = a^2 \Rightarrow |c| = |a|$$

$$\Rightarrow AB = BC.$$

Hence, $\triangle ABC$ is isosceles.

EXAMPLE 14 *In a* $\triangle ABC$, *if* $a\cos A = b\cos B$, *show that the triangle is either isosceles or right-angled.*

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = k \text{ (say)}$$

$$\Rightarrow \quad a = k\sin A \text{ and } b = k\sin B.$$

$$\therefore \quad a\cos A = b\cos B$$

$$\Rightarrow \quad k\sin A\ \cos A = k\sin B\ \cos B$$

$$\Rightarrow \quad \frac{1}{2}(\sin 2A) = \frac{1}{2}(\sin 2B)$$

$$\Rightarrow \quad \sin 2A = \sin 2B$$

$$\Rightarrow \quad \sin 2A - \sin 2B = 0$$

$$\Rightarrow \quad 2\cos(A + B)\sin(A - B) = 0$$

$$\Rightarrow \quad \cos(A + B) = 0 \quad \text{or} \quad \sin(A - B) = 0$$

$$\Rightarrow \quad (A + B) = \frac{\pi}{2} \quad \text{or} \quad (A - B) = 0$$

$$\Rightarrow \quad \angle C = \frac{\pi}{2} \quad \text{or} \quad A = B.$$

Hence, $\triangle ABC$ is right-angled or isosceles.

EXAMPLE 15 *In a* $\triangle ABC$, *if* $a = 18,\ b = 24,\ c = 30$; *find*
(i) $\sin A,\ \sin B,\ \sin C$ *(ii)* $\cos A,\ \cos B,\ \cos C$

SOLUTION Here, $a^2 + b^2 = (18)^2 + (24)^2 = (324 + 576) = 900 = (30)^2 = c^2$.

$\therefore$ $\triangle ABC$ is right-angled at C. Thus, $\angle C = 90°$.

(i) By sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

$$\Rightarrow \quad \frac{\sin A}{18} = \frac{\sin B}{24} = \frac{\sin C}{30} = k$$

$\therefore$ $\sin A = 18k$, $\sin B = 24k$ and $\sin C = 30k$.

But, $\angle C = 90^\circ \Rightarrow \sin C = \sin 90^\circ = 1 \Rightarrow 30k = 1 \Rightarrow k = \frac{1}{30}$.

$\therefore$ $\sin A = 18k = \left(18 \times \frac{1}{30}\right) = \frac{3}{5}$; $\sin B = 24k = \left(24 \times \frac{1}{30}\right) = \frac{4}{5}$

and $\sin C = 30k = \left(30 \times \frac{1}{30}\right) = 1$.

Hence, $\sin A = \frac{3}{5}$, $\sin B = \frac{4}{5}$ and $\sin C = 1$.

(ii) Using cosine formulae, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(24)^2 + (30)^2 - (18)^2}{2 \times 24 \times 30}$$

$$= \frac{576 + 900 - 324}{1440} = \frac{1152}{1440} = \frac{4}{5}.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{(30)^2 + (18)^2 - (24)^2}{2 \times 30 \times 18}$$

$$= \frac{(900 + 324 - 576)}{1080} = \frac{648}{1080} = \frac{3}{5}.$$

$\cos C = \cos 90^\circ = 0$.

Hence, $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$ and $\cos C = 0$.

EXAMPLE 16 *In a $\triangle ABC$, if $\angle A = 45^\circ$, $\angle B = 60^\circ$ and $\angle C = 75^\circ$, find the ratio of its sides.*

SOLUTION By the sine formula, we have

$$a : b : c = \sin A : \sin B : \sin C$$

$$= \sin 45^\circ : \sin 60^\circ : \sin 75^\circ$$

$$= \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{(\sqrt{3}+1)}{2\sqrt{2}} = 2 : \sqrt{6} : (\sqrt{3}+1).$$

EXAMPLE 17 *In a $\triangle ABC$, if $a = 2$, $b = 3$ and $\sin A = \left(\frac{2}{3}\right)$, find $\angle B$.*

SOLUTION $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{2}{(2/3)} = \frac{3}{\sin B}$

$$\Rightarrow \sin B = \left(3 \times \frac{2}{3} \times \frac{1}{2}\right) = 1$$

$$\Rightarrow \angle B = 90^\circ.$$

EXAMPLE 18 *The angles of a triangle ABC are in AP and if it is being given that* $b : c = \sqrt{3} : \sqrt{2}$, *find* $\angle A$, $\angle B$ *and* $\angle C$.

SOLUTION $\angle A, \angle B, \angle C$ are in AP

$\Rightarrow \quad 2\angle B = \angle A + \angle C$

$\Rightarrow \quad 3\angle B = \angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad \angle B = 60°.$

Now, $\dfrac{b}{\sin B} = \dfrac{c}{\sin C} \Rightarrow \dfrac{b}{c} = \dfrac{\sin B}{\sin C}$

$\Rightarrow \dfrac{\sqrt{3}}{\sqrt{2}} = \dfrac{\sin 60°}{\sin C}$

$\Rightarrow \sin C = \left(\dfrac{\sqrt{3}}{2} \times \sqrt{2} \times \dfrac{1}{\sqrt{3}}\right) = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \angle C = 45°.$

$\therefore \quad \angle A = 180° - (\angle B + \angle C) = [180° - (60° + 45°)] = 75°.$

Hence, $\angle A = 75°$, $\angle B = 60°$ and $\angle C = 45°$.

EXAMPLE 19 *In a* $\triangle ABC$, *if* $\angle A = 30°$ *and* $b : c = 2 : \sqrt{3}$, *find* $\angle B$.

SOLUTION Let $b = 2k$ and $c = \sqrt{3}k$. Then,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$\Rightarrow \quad \cos 30° = \dfrac{4k^2 + 3k^2 - a^2}{4\sqrt{3}\,k^2}$

$\Rightarrow \quad \dfrac{\sqrt{3}}{2} \times 4\sqrt{3}\,k^2 = 7k^2 - a^2$

$\Rightarrow \quad k^2 = a^2 \Rightarrow k = a.$

Thus, $b = 2a$ and $c = \sqrt{3}\,a$.

Now, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow \dfrac{a}{\sin 30°} = \dfrac{2a}{\sin B}$

$\Rightarrow \quad \sin B = \left(2a \times \dfrac{1}{2} \times \dfrac{1}{a}\right) = 1$

$\Rightarrow \quad B = 90°.$

EXAMPLE 20 *Solve the triangle in which* $a = (\sqrt{3} + 1)$, $b = (\sqrt{3} - 1)$ *and* $\angle C = 60°$.

SOLUTION Using $\tan \dfrac{(A - B)}{2} = \dfrac{(a - c)}{(a + b)} \cot \dfrac{C}{2}$, we get

$$\tan \frac{(A - B)}{2} = \frac{2}{2\sqrt{3}} \cot 30° = \left(\frac{1}{\sqrt{3}} \times \sqrt{3}\right) = 1$$

$\Rightarrow \quad \dfrac{A - B}{2} = 45° \Rightarrow A - B = 90°$ … (i)

Also, $A + B + C = 180° \Rightarrow A + B = 120°$ … (ii)

On solving (i) and (ii), we get $A = 105°$ and $B = 15°$.

Also, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\Rightarrow \quad \cos 60° = \dfrac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 - c^2}{2(\sqrt{3}+1)(\sqrt{3}-1)} = \dfrac{8 - c^2}{4}$

$\therefore \quad \dfrac{8 - c^2}{4} = \dfrac{1}{2} \Rightarrow 8 - c^2 = 2 \Rightarrow c^2 = 6 \Rightarrow c = \sqrt{6}.$

Hence, $c = \sqrt{6}$ cm, $\angle A = 105°$ and $\angle B = 15°$.

EXERCISE 18A

In any $\triangle ABC$, prove that

1. $a(b\cos C - c\cos B) = (b^2 - c^2)$

2. $ac\cos B - bc\cos A = (a^2 - b^2)$

3. $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{(a^2 + b^2 + c^2)}{2abc}$

4. $\dfrac{c - b\cos A}{b - c\cos A} = \dfrac{\cos B}{\cos C}$

5. $2(bc\cos A + ca\cos B + ab\cos C) = (a^2 + b^2 + c^2)$

6. $4\left(bc\cos^2\dfrac{A}{2} + ca\cos^2\dfrac{B}{2} + ab\cos^2\dfrac{C}{2}\right) = (a + b + c)^2$

7. $a\sin A - b\sin B = c\sin(A - B)$

8. $a^2\sin(B - C) = (b^2 - c^2)\sin A$

9. $\dfrac{\sin(A - B)}{\sin(A + B)} = \dfrac{(a^2 - b^2)}{c^2}$

10. $\dfrac{(b - c)}{a}\cos\dfrac{A}{2} = \sin\dfrac{(B - C)}{2}$

11. $\dfrac{(a + b)}{c}\sin\dfrac{C}{2} = \cos\dfrac{(A - B)}{2}$

12. $\dfrac{(b + c)}{a} \cdot \cos\dfrac{(B + C)}{2} = \cos\dfrac{(B - C)}{2}$

13. $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$

14. $\dfrac{(\cos^2 B - \cos^2 C)}{b + c} + \dfrac{(\cos^2 C - \cos^2 A)}{c + a} + \dfrac{(\cos^2 A - \cos^2 B)}{a + b} = 0$

15. $\dfrac{\cos 2A}{a^2} - \dfrac{\cos 2B}{b^2} = \left(\dfrac{1}{a^2} - \dfrac{1}{b^2}\right)$

16. $(c^2 - a^2 + b^2)\tan A = (a^2 - b^2 + c^2)\tan B = (b^2 - c^2 + a^2)\tan C$

17. If in a $\triangle ABC$, $\angle C = 90°$, then prove that $\sin(A - B) = \dfrac{(a^2 - b^2)}{(a^2 + b^2)}$.

18. In a $\triangle ABC$, if $\dfrac{\cos A}{a} = \dfrac{\cos B}{b}$, show that the triangle is isosceles.

19. In a $\triangle ABC$, if $\sin^2 A + \sin^2 B = \sin^2 C$, show that the triangle is right-angled.

20. Solve the triangle in which $a = 2$ cm, $b = 1$ cm and $c = \sqrt{3}$ cm.

21. In a $\triangle ABC$, if $a = 3$ cm, $b = 5$ cm and $c = 7$ cm, find $\cos A$, $\cos B$, $\cos C$

22. If the angles of a triangle are in the ratio $1:2:3$, prove that its corresponding sides are in the ratio $1:\sqrt{3}:2$.

ANSWERS (EXERCISE 18A)

20. $\angle A = 90°, \angle B = 30°$ and $\angle C = 60°$

21. $\cos A = \frac{13}{14}$, $\cos B = \frac{11}{14}$ and $\cos C = \frac{-1}{2}$

HINTS TO SOME SELECTED QUESTIONS

6. $\text{LHS} = 2[bc(1+\cos A) + ca(1+\cos B) + ab(1+\cos C)]$

$= 2[(bc + ca + ab) + bc \cdot \cos A + ca \cdot \cos B + ab \cdot \cos C]$

$= 2\left[(ab + bc + ca) + bc \cdot \frac{(b^2 + c^2 - a^2)}{2bc} + ca \cdot \frac{(c^2 + a^2 - b^2)}{2ca} + ab \cdot \frac{(a^2 + b^2 - c^2)}{2ab}\right]$

$= a^2 + b^2 + c^2 + 2(ab + bc + ca) = (a + b + c)^2 = \text{RHS}.$

7. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\Rightarrow \quad a = k \sin A, b = k \sin B$ and $c = k \sin C$.

$\therefore \quad \text{LHS} = k \sin^2 A - k \sin^2 B$

$= k(\sin^2 A - \sin^2 B) = k \sin (A + B) \cdot \sin (A - B)$

$= k \sin (\pi - C) \sin (A - B) = k \sin C \sin (A - B)$

$= c \sin (A - B) = \text{RHS}.$

8. $\text{LHS} = k^2 \sin^2 A[\sin (B - C)]$

$= k^2 \sin A \sin [\pi - (B + C)] \sin (B - C)$

$= k^2 \sin A \sin (B + C) \sin (B - C)$

$= k^2 \sin A \cdot (\sin^2 B - \sin^2 C) = \text{RHS}.$

9. See Example 3.

For questions **10, 11, 12** see Example 4.

13. $a^2(\cos^2 B - \cos^2 C) = a^2[(1 - \sin^2 B) - (1 - \sin^2 C)] = a^2(\sin^2 C - \sin^2 B)$

$= a^2\left(\frac{c^2}{k^2} - \frac{b^2}{k^2}\right) = \frac{1}{k^2} \cdot a^2(c^2 - b^2).$

14. $\frac{\cos^2 B - \cos^2 C}{(b + c)} = \frac{(1 - \sin^2 B) - (1 - \sin^2 C)}{(b + c)}$

$= \frac{(\sin^2 C - \sin^2 B)}{(b + c)} = \frac{(c^2 - b^2)}{k^2(b + c)} = \frac{1}{k^2}(c - b).$

15. $\text{LHS} = \frac{(1 - 2\sin^2 A)}{a^2} - \frac{(1 - 2\sin^2 B)}{b^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right) - \frac{2\sin^2 A}{a^2} + \frac{2\sin^2 B}{b^2}$

$= \left(\frac{1}{a^2} - \frac{1}{b^2}\right) - \frac{2\sin^2 A}{k^2 \sin^2 A} + \frac{2\sin^2 B}{k^2 \sin^2 B}$

$= \left(\frac{1}{a^2} - \frac{1}{b^2}\right) - \frac{2}{k^2} + \frac{2}{k^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right).$

17. $\text{RHS} = \dfrac{(a^2 - b^2)}{(a^2 + b^2)}$

$$= \frac{k^2 \sin^2 A - k^2 \sin^2 B}{k^2 \sin^2 A + k^2 \sin^2 B} = \frac{(\sin^2 A - \sin^2 B)}{\sin^2 A + \sin^2 (90^\circ - A)} = (\sin^2 A - \sin^2 B)$$

$$= \sin (A + B) \sin (A - B) = (\sin 90^\circ) \sin (A - B) = \sin (A - B).$$

18. $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} \Rightarrow \dfrac{(b^2 + c^2 - a^2)}{2abc} = \dfrac{a^2 + c^2 - b^2}{2abc}$

$$\Rightarrow b^2 + c^2 - a^2 = a^2 + c^2 - b^2 \Rightarrow 2a^2 = 2b^2$$

$$\Rightarrow a^2 = b^2 \Rightarrow a = b.$$

19. $\sin^2 A + \sin^2 B = \sin^2 C \Rightarrow \dfrac{a^2}{k^2} + \dfrac{b^2}{k^2} = \dfrac{c^2}{k^2}$ $\left[\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k \text{ (say)}\right]$

$$\Rightarrow a^2 + b^2 = c^2 \Rightarrow \triangle ABC \text{ is right-angled.}$$

22. Let $\angle A = x^\circ$, $\angle B = (2x)^\circ$ and $\angle C = (3x)^\circ$. Then,

$A + B + C = 180^\circ \Rightarrow x + 2x + 3x = 180 \Rightarrow 6x = 180 \Rightarrow x = 30.$

$\therefore \quad \angle A = 30^\circ, \angle B = 60^\circ$ and $\angle C = 90^\circ$.

Ratio of sides $= \sin A : \sin B : \sin C = \sin 30^\circ : \sin 60^\circ : \sin 90^\circ$

$$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2.$$

PROBLEMS BASED ON SINE AND COSINE FORMULAE

EXAMPLE 1 *Two ships leave a port at the same time. One goes* 24 km *per hour in the direction* N 45° E *and the other travels* 32 km *per hour in the direction* S 75° E. *Find the distance between the ships at the end of* 3 *hours.*

SOLUTION Let A and B be the positions of the two ships at the end of 3 hours.

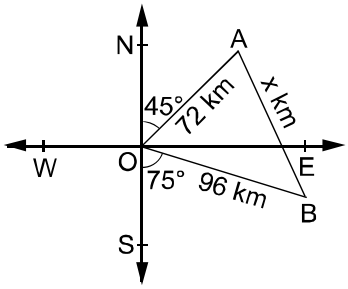
Then, $OA = (24 \times 3)$ km $= 72$ km.

And, $OB = (32 \times 3)$ km $= 96$ km.

Let, $AB = x$ km.

We have: $\angle NOA = 45^\circ$ and $\angle SOB = 75^\circ$

$\therefore \quad \angle AOB = 180^\circ - (45^\circ + 75^\circ) = 60^\circ.$



Applying cosine formula on $\triangle AOB$, we get

$$\cos 60^\circ = \frac{(72)^2 + (96)^2 - x^2}{2 \times 72 \times 96}$$

$$\Rightarrow \quad (72)^2 + (96)^2 - x^2 = \frac{1}{2} \times 2 \times 72 \times 96$$

$$\Rightarrow \quad 5184 + 9216 - x^2 = 6912$$

$$\Rightarrow \quad x^2 = 14400 - 6912 = 7488$$

$$\Rightarrow \quad x = \sqrt{7488} = 86.53 \text{ km}.$$

Hence, the distance between the ships at the end of 3 hours is 86.53 km.

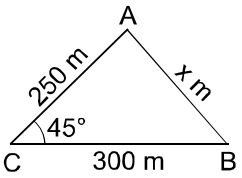
EXAMPLE 2 *Two trees A and B are on the same side of a river. From a point C in the river, the distances of trees A and B are* 250 m *and* 300 m *respectively. If* $\angle C = 45°$, *find the distance between the trees.* (*Use* $\sqrt{2} = 1.44$.)

SOLUTION Let A and B be the trees and C be a point in the river.

Then, $CA = 250$ m, $CB = 300$ m and $\angle ACB = 45°$.

Let $AB = x$ metres.

Applying cosine formula on $\triangle ACB$, we get



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow \quad \cos 45° = \frac{(300)^2 + (250)^2 - x^2}{2 \times 300 \times 250}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{90000 + 62500 - x^2}{150000}$$

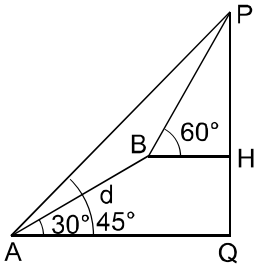
$$\Rightarrow \quad \frac{152500 - x^2}{150000} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad 152500 - x^2 = 75000 \times \sqrt{2} = 75000 \times 1.44 = 108000$$

$$\Rightarrow \quad x^2 = 152500 - 108000 = 44500$$

$$\Rightarrow \quad x = \sqrt{44500} = 210.95 \text{ m}.$$

EXAMPLE 3 *The angle of elevation of the top point P of the vertical tower PQ of height h from a point A on the ground is* 45° *and from a point B, the angle of elevation is* 60°, *where B is a point at a distance d from the point A, measured along the line AB, which makes an angle* 30° *with AQ. Prove that* $d = h(\sqrt{3} - 1)$.



SOLUTION In $\triangle PAQ$, we have

$$\frac{AQ}{PQ} = \cot 45° = 1 \;\Rightarrow\; \frac{AQ}{h} = 1 \;\Rightarrow\; AQ = h.$$

From right $\triangle AQP$, we get

$$AP^2 = AQ^2 + PQ^2 = (h^2 + h^2) = 2h^2 \;\Rightarrow\; AP = \sqrt{2}h.$$

In $\triangle PBH$, $\angle BPH = 180° - (60° + 90°) = 30°$.

In $\triangle PAQ$, $\angle APQ = 180° - (45° + 90°) = 45°$.

$\therefore \quad \angle APB = (\angle APQ - \angle BPH) = (45° - 30°) = 15°$.

In $\triangle APB$, we have

$\angle PAB = 15°$, $\angle APB = 15°$ and $\angle ABP = 180° - (15° + 15°) = 150°$.

Using sine formula on $\triangle ABP$, we have

$$\frac{AB}{\sin 15°} = \frac{AP}{\sin 15°}$$

$$\Rightarrow \quad \frac{d}{\sin 15°} = \frac{\sqrt{2}h}{\sin 30°}$$

$$\Rightarrow \quad d = 2\sqrt{2}h \cdot \sin 15^\circ$$

$$\Rightarrow \quad d = 2\sqrt{2}h \cdot \frac{(\sqrt{3}-1)}{2\sqrt{2}}$$

$$= (\sqrt{3}-1)h.$$

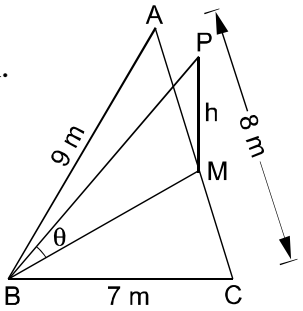
Hence, $d = (\sqrt{3}-1)h.$

$$\left[\because \ \sin 15^\circ = (\sin 45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.\right]$$

EXAMPLE 4 *A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with BC = 7 m, CA = 8 m and AB = 9 m. The lamp post subtends an angle* $\tan^{-1} 3$ *at the point B. Determine the height of the lamp post.*

SOLUTION We have:

$a = BC = 7$ m, $b = AC = 8$ m and $C = AB = 9$ m.



Let M be the midpoint of AC and let MP be the lamp post of height h metres.

Then, $\angle MBP = \theta = \tan^{-1}\frac{1}{3}$.

Applying cosine formula on $\triangle ABC$, we get

$$\cos C = \frac{(a^2 + b^2 - c^2)}{2ab} = \frac{(49 + 64 - 81)}{2 \times 7 \times 8} = \frac{32}{112} = \frac{2}{7} \qquad \ldots \text{(i)}$$

In $\triangle BMC$, we have

$$BC = 7 \text{ m}, CM = \frac{1}{2}CA = 4 \text{ m}.$$

$$\therefore \quad BM^2 = BC^2 + CM^2 - 2BC \times CM \times \cos C$$

$$= \left(49 + 16 - 2 \times 7 \times 4 \times \frac{2}{7}\right) = 49 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad BM = 7 \text{ m}.$$

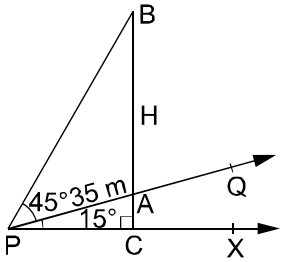
From right $\triangle BMP$, we have

$$\tan\theta = \frac{PM}{BM} = \frac{h}{7} \Rightarrow \frac{h}{7} = \tan(\tan^{-1} 3) = 3 \Rightarrow h = 21 \text{ m}.$$

Hence, the height of the lamp post is 21 m.

EXAMPLE 5 *A tree stands vertically on a hill side which makes an angle of* 15° *with the horizontal. From a point on the ground* 35 m *down the hill from the base of the tree, the angle of elevation of the top of tree is* 60°. *Find the height of the tree.*

SOLUTION Let PAQ be the hill, AB be the tree and PCX be the horizontal. Let P be the point of observation. Produce BA to meet PX at C. Let $AB = H$ metres.



Then, $\angle XPA = 15^\circ$, $PA = 35$ m,

$\angle CPB = 60^\circ$ and $\angle PCA = 90^\circ$.

$\therefore \quad \angle APB = (60^\circ - 15^\circ) = 45^\circ$.

In $\triangle PAC$, $\angle PAC = 180^\circ - (15^\circ + 90^\circ) = 75^\circ$

$\therefore \quad \angle PAB = (180° - 75) = 105°.$

And, $\angle PBA = 180° - (45° + 105°) = 30°.$

Applying sine rule on $\triangle PAB$, we get

$$\frac{PA}{\sin \angle PBA} = \frac{AB}{\sin \angle APB} \Rightarrow \frac{35}{\sin 30°} = \frac{H}{\sin 45°}$$

$\therefore \quad 35 \times 2 = H \times \sqrt{2} \Rightarrow H = 35\sqrt{2}$ m.

Hence, the height of the tree is $35\sqrt{2}$ m.

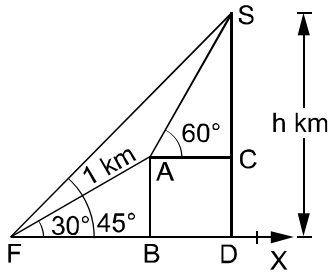
EXERCISE 18B

1. Two boats leave a port at the same time. One travels 60 km in the direction N 50° E while the other travels 50 km in the direction S 70° E. What is the distance between the boats?

2. A town B is 12 km south and 18 km west of a town A. Show that the bearing of B from A is S $56° 20'$ W. Also, find the distance of B from A.

[Given: $\tan 56°20' = 1.5$.]

3. At the foot of a mountain, the angle of elevation of its summit is 45°. After ascending 1 km towards the mountain up an incline of 30°, the elevation changes to 60° (as shown in the given figure). Find the height of the mountain. [Given: $\sqrt{3} = 1.73$.]



ANSWERS (EXERCISE 18B)

1. 55.68 km **2.** 21.63 km **3.** 1.365 km

HINTS TO SOME SELECTED QUESTIONS

1. Let the original position of each boat be at O. Let A and B be their respective final positions, as shown in the figure.

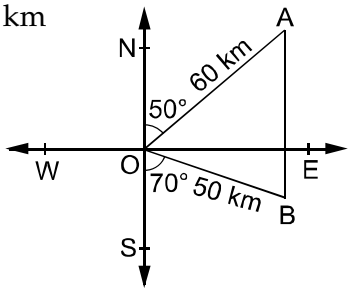
Then, $\angle NOA = 50°$, $\angle SOB = 70°$, $OA = 60$ km and $OB = 50$ km

$\therefore \quad \angle AOB = 180° - (50° + 70°) = 60°$. Let $AB = x$ km.

By cosine formula on $\triangle AOB$, we have

$$\cos \angle AOB = \frac{OA^2 + OB^2 - AB^2}{2 \times OA \times OB}$$

$$\Rightarrow \quad \cos 60° = \frac{(60)^2 + (50)^2 - x^2}{2 \times 60 \times 50}$$



$$\Rightarrow \quad 3600 + 2500 - x^2 = \frac{1}{2} \times 2 \times 60 \times 50 \Rightarrow x^2 = 3100 \Rightarrow x = \sqrt{3100} = 55.68 \text{ km}.$$

Hence, the required distance between the boats is 55.68 km.

2. Let B be the position of the town relative to A, as shown in the figure.

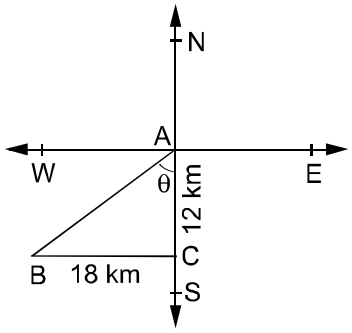
From B, draw $BC \perp AS$.

Then, $AC = 12$ km and $BC = 18$ km.

Let $\angle BAC = \theta$. Then,

$$\tan\theta = \frac{BC}{AC} = \frac{18}{12} = \frac{3}{2} \Rightarrow \theta = 56°20'.$$

$\therefore$ bearing of B from A is S 56°20′ W.



Also, $AB = \sqrt{BC^2 + AC^2} = \sqrt{(18)^2 + (12)^2} = \sqrt{324 + 144}$

$= \sqrt{468} = 21.63$ km.

Hence, the distance of B from A is 21.63 km.

3. $\frac{SD}{FD} = \tan 45° = 1 \Rightarrow SD = FD = h$ km

$\Rightarrow \angle FSD = \angle SFD = 45°.$

$\angle ASC = 180° - (60° + 90°) = 30°$

$\Rightarrow \angle FSA = (45° - 30°) = 15°$ and $\angle SFA = 15°$.

$\therefore \angle SAF = 180° - (15° + 15°) = 150°.$

By the sine formula on $\triangle SAF$, we have

$$\frac{AF}{\sin\angle FSA} = \frac{FS}{\sin\angle SAF} \Rightarrow \frac{1}{\sin 15°} = \frac{\sqrt{2}\,h}{\sin 150°} = \frac{\sqrt{2}\,h}{\sin 30°} \Rightarrow \frac{2\sqrt{2}}{(\sqrt{3}-1)} = 2\sqrt{2}h$$

$$\left[\because FS = \sqrt{h^2 + h^2} = \sqrt{2}h \text{ and } \sin 15° = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.\right]$$

$$\Rightarrow h = \frac{1}{(\sqrt{3}-1)} \times \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{(\sqrt{3}+1)}{2} = \frac{(1.73+1)}{2} = \frac{2.73}{2} = 1.365 \text{ km.}$$

19

Graphs of Trigonometric Functions

GRAPH OF $y = \sin x$

We know that sin x is periodic with period 2π.

So, we draw the graph of sin x in the interval $[0, 2\pi]$.

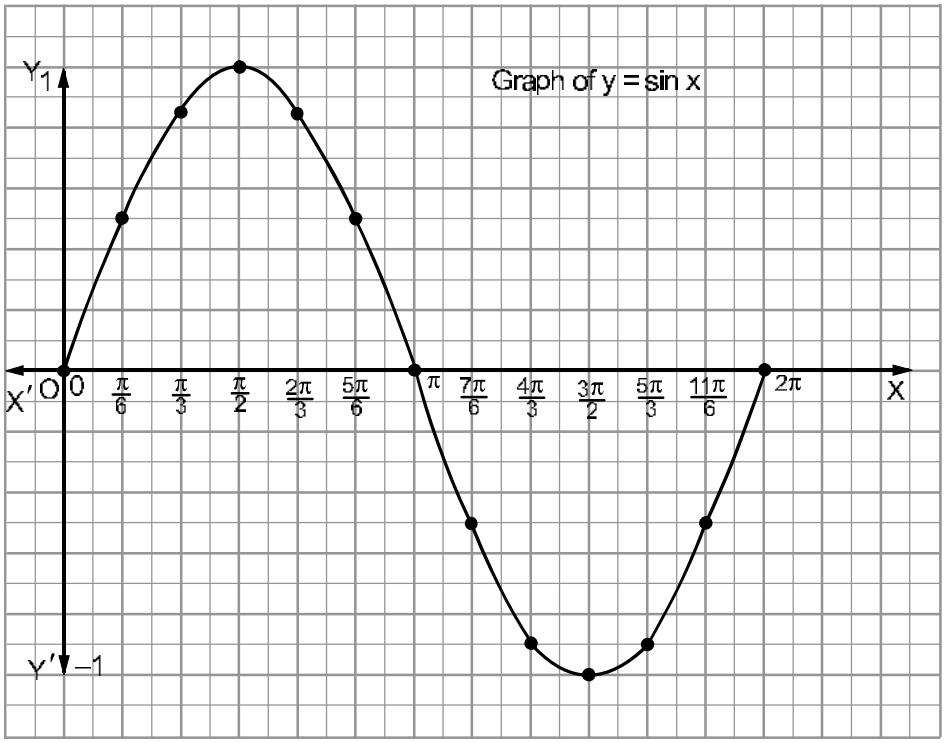
We can then extend it easily by repeating it over intervals of length 2π.

For this, we first construct the table given below.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
$\sin x$	0	$\frac{1}{2}$ $= 0.5$	$\frac{\sqrt{3}}{2}$ $= 0.87$	1	$\frac{\sqrt{3}}{2}$ $= 0.87$	$\frac{1}{2}$ $= 0.5$	0	$\frac{-1}{2}$ $= -0.5$	$\frac{-\sqrt{3}}{2}$ $= -0.87$	-1	$\frac{-\sqrt{3}}{2}$ $= -0.87$	$\frac{-1}{2}$ $= -0.5$	0

On a graph paper we take 1 cm as $\pi/6$ units on the x-axis, and 5 cm as 1 unit on the y-axis. Taking the values of x along the x-axis and the values of sin x along the y-axis, we plot the points $(0, 0)$, $(\pi/6, 0.5)$, $(\pi/3, 0.87)$, $(\pi/2, 1)$, $(2\pi/3, 0.87)$, $(5\pi/6, 0.5)$, $(\pi, 0)$, $(7\pi/6, -0.5)$, $(4\pi/3, -0.87)$, $(3\pi/2, -1)$, $(5\pi/3, -0.87)$, $(11\pi/6, -0.5)$ and $(2\pi, 0)$.

We join these points by a freehand curve and obtain the graph of $y = \sin x$, as shown below.



GRAPH OF $y = \cos x$

We know that $\cos x$ is periodic with period 2π.

So, we draw the graph of $\cos x$ in the interval $[0, 2\pi]$.

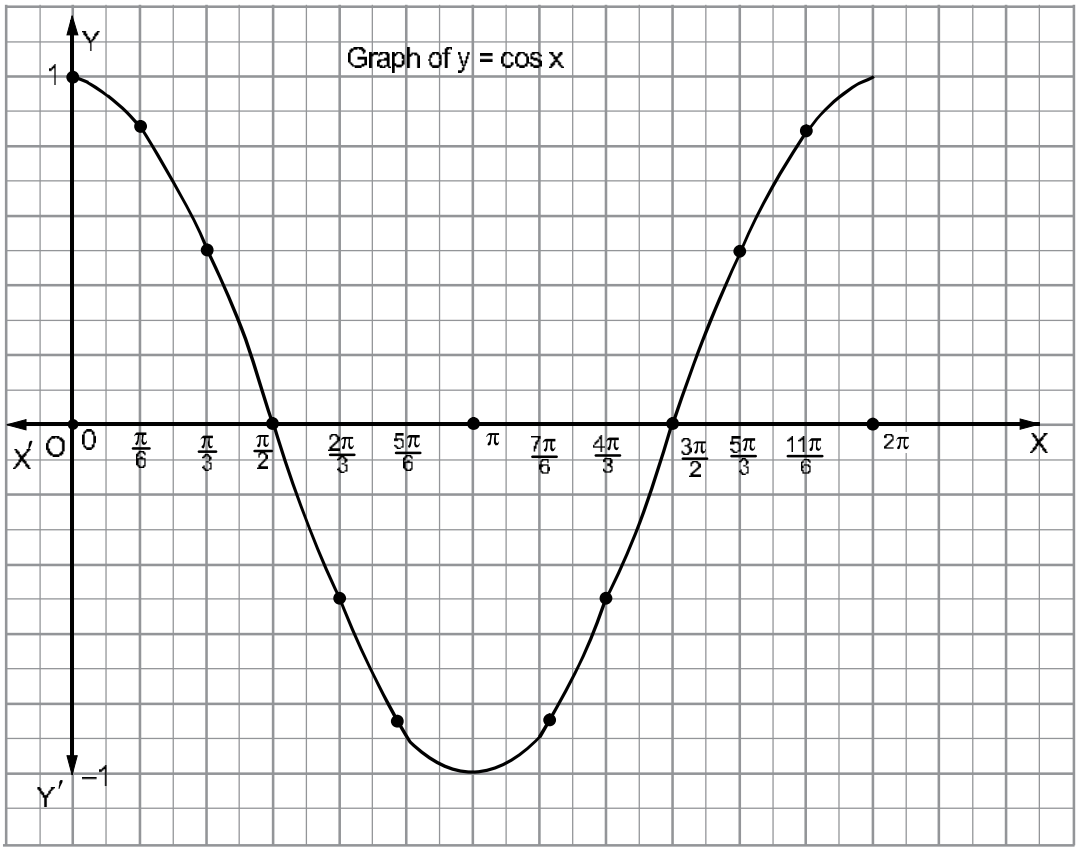
We can then extend it easily by repeating it over intervals of length 2π.

For this, we first construct the table given below.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$ = 0.87	$\frac{1}{2}$ = 0.5	0	$\frac{-1}{2}$ = –0.5	$\frac{-\sqrt{3}}{2}$ = –0.87	–1	$\frac{-\sqrt{3}}{2}$ = –0.87	$\frac{-1}{2}$ = –0.5	0	$\frac{1}{2}$ = 0.5	$\frac{\sqrt{3}}{2}$ = 0.87	1

On a graph paper, we take 1 cm as $\pi/6$ units on the x-axis, and 5 cm as 1 unit on the y-axis.

Taking the values of x along the x-axis and the values of $\cos x$ along the y-axis, we plot the points $(0, 1)$, $(\pi/6, 0.87)$, $(\pi/3, 0.5)$, $(\pi/2, 0)$, $(2\pi/3, -0.5)$, $(5\pi/6, -0.87)$, $(\pi, -1)$, $(7\pi/6, -0.87)$, $(4\pi/3, -0.5)$, $(3\pi/2, 0)$, $(5\pi/3, 0.5)$, $(11\pi/6, 0.87)$ and $(2\pi, 1)$ on the graph paper and, joining these points successively by a freehand curve, we obtain the graph of $\cos x$, as shown below.



GRAPH OF $y = \tan x$

We know that $y = \tan x$ is defined for all real values of x, except when x is an odd multiple of $\pi/2$.

Also, $\tan x$ is periodic with period π.

So, we draw the graph of $y = \tan x$ in $[0, \pi]$ and then extend it over other intervals.

As x increases from 0 to $\pi/2$, $\tan x$ increases from 0 to ∞.

As x crosses the value of $\pi/2$, $\tan x$ becomes negative and is arbitrarily large in magnitude.

As x approaches $\pi/2$ from π, $\tan x$ comes to 0.

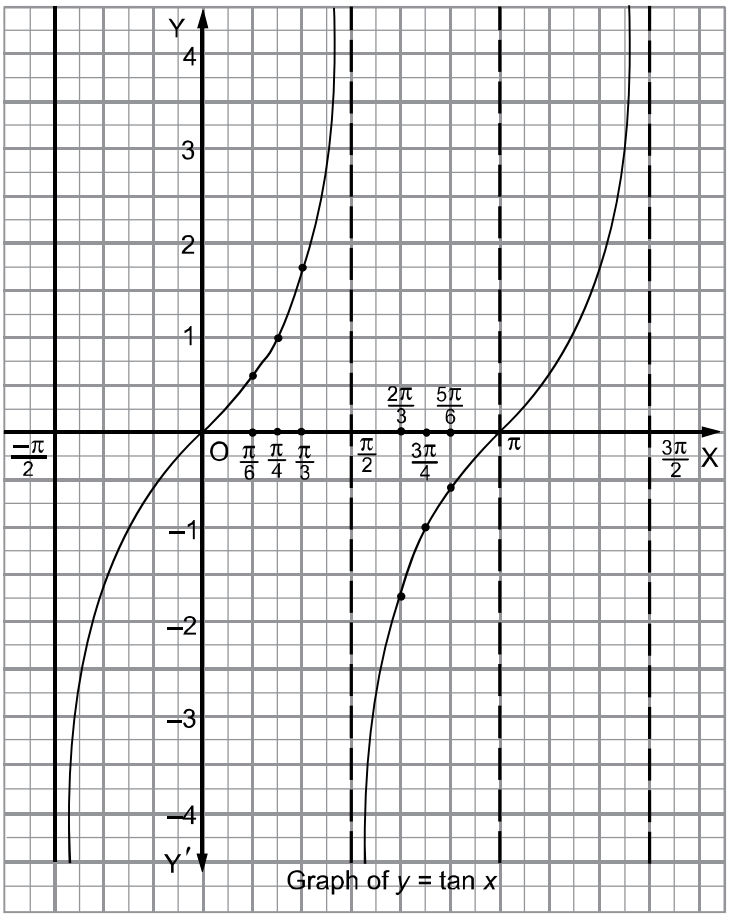
Now, we can construct the table, given below.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan x$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $= \frac{1.73}{3}$ $= 0.58$	1	$\sqrt{3}$ $= 1.73$	∞	$-\sqrt{3}$ $= -1.73$	-1	$-\frac{1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$ $= \frac{-1.73}{3}$ $= -0.58$	0

On a graph paper, we take 1 cm as $\pi/6$ units on the x-axis, and 2 cm as 1 unit on the y-axis.

Taking the values of x on the x-axis and the values of $\tan x$ along the y-axis, we plot the points $(0, 0)$, $(\pi/6, 0.58)$, $(\pi/4, 1)$, $(\pi/3, 1.73)$, $(\pi/2, \infty)$, $(2\pi/3, -1.73)$, $(3\pi/4, -1)$, $(5\pi/6, -0.58)$ and $(\pi, 0)$.

We join these points by a freehand curve and obtain the graph of $\tan x$ as shown below.



Graph of $y = \tan x$

GRAPH OF $y = a \sin bx$

We know that $-1 \le \sin bx \le 1$.

$\therefore \quad -a \le a \sin bx \le a$.

Also, $\sin x$ is periodic with period 2π.

$\therefore$ $\sin bx$ is periodic with period $\dfrac{2\pi}{b}$.

Hence, $a \sin bx$ is periodic with period $\dfrac{2\pi}{b}$.

EXAMPLE *Draw the graph of* $y = 3 \sin 2x$.

SOLUTION $-1 \le \sin 2x \le 1 \Rightarrow -3 \le 3 \sin 2x \le 3$.

Also, $3 \sin 2x$ is periodic with period $\dfrac{2\pi}{2}$, i.e., π.

So, we draw the graph of $y = 3 \sin 2x$ in the interval $[0, \pi]$.

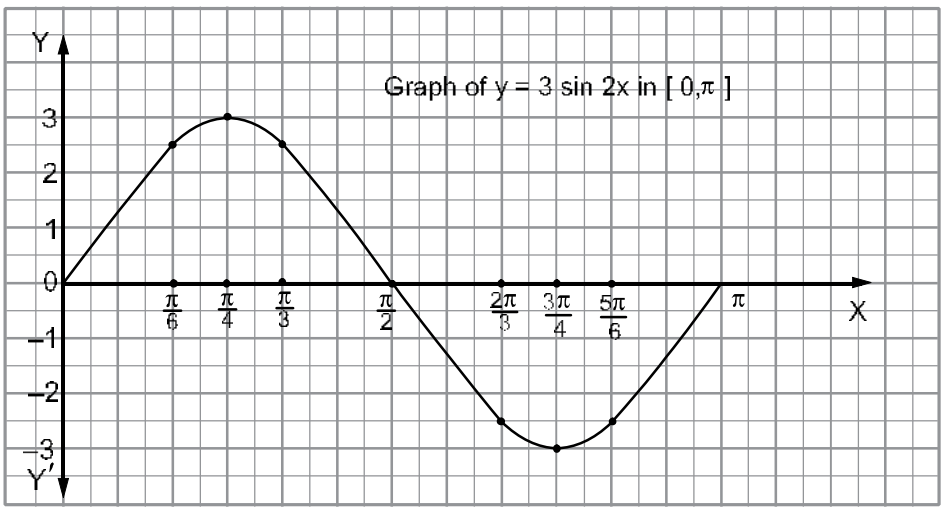
We thus prepare the table given below.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$2x$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin 2x$	0	$\frac{\sqrt{3}}{2}$ $= 0.87$	1	$\frac{\sqrt{3}}{2}$ $= 0.87$	0	$\frac{-\sqrt{3}}{2}$ $= -0.87$	–1	–0.87	0
$3 \sin 2x$	0	2.61	3	2.61	0	–2.61	–3	–2.61	0

On a graph paper, we take 4 cm = $\pi/3$ units on the x-axis and 1 cm = 1 unit on the y-axis.

Now, we plot the points $(0, 0)$, $(\pi/6, 2.61)$, $(\pi/4, 3)$, $(\pi/3, 2.61)$, $(\pi/2, 0)$, $(2\pi/3, -2.61)$, $(3\pi/4, -3)$, $(5\pi/6, -2.61)$, and $(\pi, 0)$.

Join these points with a freehand curve to obtain the graph of $y = 3 \sin 2x$, given below.



GRAPH OF $y = a \cos bx$

We know that $-1 \le \cos bx \le 1$.

$\therefore \quad -a \le a \cos bx \le a$.

Also, $\cos x$ is periodic with period 2π.

$\therefore$ $\cos bx$ is periodic with period $\dfrac{2\pi}{b}$.

Hence, $a \cos bx$ is periodic with period $\dfrac{2\pi}{b}$.

EXAMPLE 1 *Draw the graph of* $y = 3 \cos 2x$.

SOLUTION $-1 \le \cos 2x \le 1 \Rightarrow -3 \le 3 \cos 2x \le 3$.

Also, $3 \cos 2x$ is periodic with period $\dfrac{2\pi}{2}$, i.e., π.

So, we draw the graph of $y = 3 \cos 2x$ in the interval $[0, \pi]$.

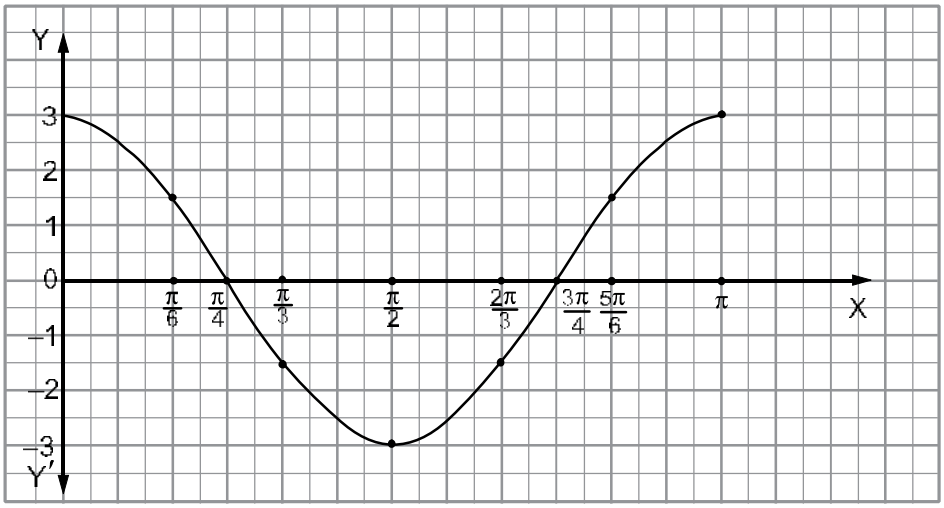
We thus prepare the table given below.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$2x$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos 2x$	1	$\frac{1}{2}$ $= 0.5$	0	$-\frac{1}{2}$ $= -0.5$	-1	$-\frac{1}{2}$ $= -0.5$	0	$\frac{1}{2}$ $= 0.5$	1
$3 \cos 2x$	3	1.5	0	-1.5	-3	-1.5	0	1.5	3

On a graph paper, we take 4 cm as $\pi/3$ units on the x-axis, and 1 cm = 1 unit on the y-axis.

Now, plot the points $(0, 3)$, $(\pi/6, 1.5)$, $(\pi/4, 0)$, $(\pi/3, -1.5)$, $(\pi/2, -3)$, $(2\pi/3, -1.5)$, $(3\pi/4, 0)$, $(5\pi/6, 1.5)$ and $(\pi, 3)$.

Join them with a freehand curve to obtain the required graph, given below.



EXAMPLE 2 *Draw the graphs of* $y = \cos x$ *and* $y = \cos \frac{1}{2} x$ *in* $[0, 2\pi]$ *on the same scale.*

SOLUTION We know that

$y = \cos x$ is a periodic function with period 2π and amplitude 1

$y = \cos \frac{1}{2} x$ is a periodic function with period 4π and amplitude 1.

We may prepare the table as under:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0	0.5	0.87	1
$\frac{1}{2} x$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\cos \frac{1}{2} x$	1	0.97	0.87	0.7	0.5	0.26	0	–0.26	–0.5	–0.7	–0.87	–0.97	–1

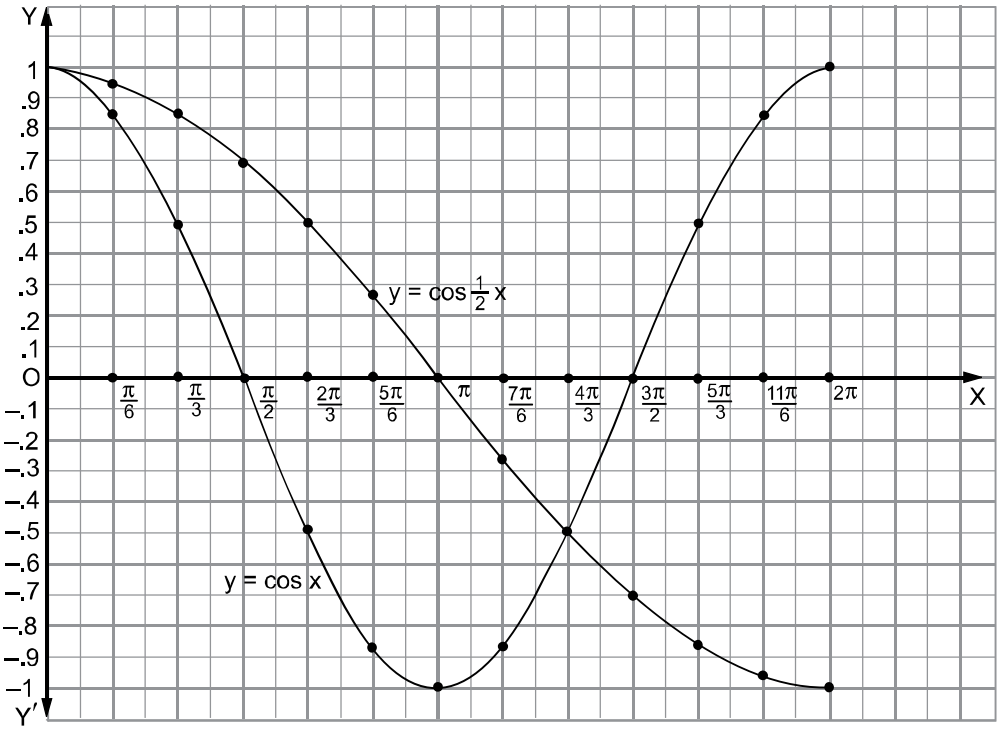
On a graph paper, we take 1 cm = $\pi/6$ units on the x-axis, and 0.5 cm = 0.1 unit on the y-axis.

For $y = \cos x$, we plot the points $(0, 1)$, $(\pi/6, 0.87)$, $(\pi/3, 0.5)$, $(\pi/2, 0)$, $(2\pi/3, -0.5)$, $(5\pi/6, -0.87)$, $(\pi, -1)$, $(7\pi/6, -0.87)$, $(4\pi/3, -0.5)$, $(3\pi/2, 0)$, $(5\pi/3, 0.5)$, $(11\pi/6, 0.87)$ and $(2\pi, 1)$.

We join these points by a freehand curve to obtain the curve $y = \cos x$.

For $y = \cos \frac{1}{2} x$, we plot the points $(0, 1)$, $(\pi/6, 0.97)$, $(\pi/3, 0.87)$, $(\pi/2, 0.7)$, $(2\pi/3, 0.5)$, $(5\pi/6, 0.26)$, $(\pi, 0)$, $(7\pi/6, -0.26)$, $(4\pi/3, -0.5)$, $(3\pi/2, -0.7)$, $(5\pi/3, -0.87)$, $(11\pi/6, -0.97)$ and $(2\pi, -1)$.

We join these points by a freehand curve to obtain the curve $y = \cos \frac{1}{2} x$.



EXERCISE 19

Draw the graph of each of the following functions:

1. $\sin 3x$ **2.** $3\sin x$ **3.** $2\sin 3x$

4. $2\cos 3x$ **5.** $\sin \frac{x}{2}$

6. Draw the graphs of $y = \sin x$ and $y = \cos x$ in $[0, 2\pi]$ on the same axes.

7. Draw the graphs of $y = \cos x$ and $y = \cos 2x$ in $[0, 2\pi]$ on the same axes.

4

Principle of Mathematical Induction

PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(i) *$P(1)$ is true and* (ii) *$P(k+1)$ is true, whenever $P(k)$ is true*

then $P(n)$ is true for all $n \in N$.

SOLVED EXAMPLES

EXAMPLE 1 *Using the principle of mathematical induction, prove that*

$1+3+5+7+\ldots+(2n-1)=n^2$ *for all* $n \in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n):\ 1+3+5+7+\ldots+(2n-1)=n^2$.

Putting $n=1$ in the given statement, we get

LHS = 1 and RHS = $1^2=1$.

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k):\ 1+3+5+7+\ldots+(2k-1)=k^2$. ... (i)

Now, $1+3+5+7+\ldots+(2k-1)+\{2(k+1)-1\}$

$=\{1+3+5+7+\ldots+(2k-1)\}+(2k+1)$

$=k^2+(2k+1)$ [using (i)]

$=(k+1)^2$.

$\therefore$ $P(k+1):\ 1+3+5+7+\ldots(2k-1)+\{2(k+1)-1\}=(k+1)^2$.

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$1+3+5+7+\ldots+(2n-1)=n^2$ for all $n \in N$.

EXAMPLE 2 *Using the principle of mathematical induction, prove that*

$1+4+7+10+\ldots+(3n-2)=\frac{1}{2}n(3n-1)$ *for all* $n \in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n):\ 1+4+7+10+\ldots+(3n-2)=\frac{1}{2}n(3n-1)$.

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = 1 \text{ and RHS} = \frac{1}{2} \times 1 \times (3 \times 1 - 1) = 1.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): \ 1 + 4 + 7 + 10 + \ldots + (3k - 2) = \frac{1}{2}k(3k - 1). \qquad \ldots \text{(i)}$$

Now, $1 + 4 + 7 + \ldots + (3k - 2) + \{3(k + 1) - 2\}$

$$= \{1 + 4 + 7 + \ldots (3k - 2)\} + (3k + 1)$$

$$= \frac{1}{2}k(3k - 1) + (3k + 1) \qquad \text{[using (i)]}$$

$$= \frac{1}{2}(3k^2 - k + 6k + 2) = \frac{1}{2}(3k^2 + 5k + 2)$$

$$= \frac{1}{2}(3k^2 + 3k + 2k + 2) = \frac{1}{2} \cdot \{3k(k + 1) + 2(k + 1)\}$$

$$= \frac{1}{2}(k + 1)(3k + 2) = \frac{1}{2}(k + 1)\{3(k + 1) - 1\}.$$

$\therefore$ $P(k+1): \ 1 + 4 + 7 + \ldots + \{3(k + 1) - 2\} = \frac{1}{2}(k + 1)\{3(k + 1) - 1\}.$

This shows that $P(k + 1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1 + 4 + 7 + \ldots + (3n - 2) = \frac{1}{2}n(3n - 1) \text{ for all } n \in N.$$

EXAMPLE 3 *Using the principle of mathematical induction, prove that*

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1) \text{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n): \ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1).$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = 1^2 = 1 \text{ and RHS} = \frac{1}{6} \times 1 \times 2 \times (2 \times 1 + 1) = 1.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): \ 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k + 1)(2k + 1). \qquad \ldots \text{(i)}$$

Now, $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2$

$$= \{1^2 + 2^2 + 3^2 + \ldots + k^2\} + (k + 1)^2$$

$$= \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2 \qquad \text{[using (i)]}$$

$$= \frac{1}{6}(k+1)\cdot\{k(2k+1)+6(k+1)\}$$

$$= \frac{1}{6}(k+1)(2k^2+7k+6) = \frac{1}{6}(k+1)\{(2k^2+4k)+(3k+6)\}$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(k+1+1)[2(k+1)+1].$$

$\therefore$ $P(k+1)$: $1^2+2^2+3^2+\ldots+(k+1)^2$

$$= \frac{1}{6}(k+1)(k+1+1)[2(k+1)+1].$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1) \text{ for all } n \in N.$$

EXAMPLE 4 *Using the principle of mathematical induction, prove that*

$$1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 \text{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = 1^3 = 1 \text{ and RHS} = \left(\frac{1\times 2}{2}\right)^2 = 1^2 = 1.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true for some $k \in N$. Then,

$$P(k):\ 1^3+2^3+3^3+\ldots k^3 = \left\{\frac{k(k+1)^2}{2}\right\}^2. \qquad \ldots \text{(i)}$$

Now, $1^3+2^3+3^3+\ldots+k^3+(k+1)^3$

$$= \{1^3+2^3+3^3+\ldots+k^3\}+(k+1)^3$$

$$= \left\{\frac{k(k+1)}{2}\right\}^2+(k+1)^3 \qquad \text{[using (i)]}$$

$$= (k+1)^2\left\{\frac{k^2}{4}+(k+1)\right\} = (k+1)^2\left\{\frac{k^2+4k+4}{4}\right\}$$

$$= \frac{(k+1)^2(k+2)^2}{4} = \left\{\frac{(k+1)\{(k+1)+1\}}{2}\right\}^2$$

$\therefore$ $P(k+1)$: $1^3+2^3+3^3+\ldots+(k+1)^3 = \left[\frac{(k+1)\{(k+1)+1\}}{2}\right]^2$.

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1^3+2^3+3^3+\ldots+n^3=\left\{\frac{n(n+1)}{2}\right\}^2 \text{ for all } n\in N.$$

EXAMPLE 5 *Using the principle of mathematical induction, prove that*

$$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2).$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2).$$

When $n=1$, LHS $=1\cdot 2=2$ and RHS $=\frac{1}{3}\times 1\times 2\times(1+2)=2.$

$\therefore$ LHS = RHS.

Thus, the given statement is true for $n=1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ 1\cdot 2+2\cdot 3+3\cdot 4+\ldots+k(k+1)=\frac{1}{3}k(k+1)(k+2). \quad \ldots \text{(i)}$$

Now, $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+k(k+1)+(k+1)(k+2)$

$=\{1\cdot 2+2\cdot 3+3\cdot 4+\ldots+k(k+1)+(k+1)(k+2)$

$=\frac{1}{3}k(k+1)(k+2)+(k+1)(k+2)$ [using (i)]

$=\frac{1}{3}\cdot[k(k+1)(k+2)+3(k+1)(k+2)]=\frac{1}{3}(k+1)(k+2)(k+3).$

$\therefore$ $P(k+1):\ 1\cdot 2+2\cdot 3+3\cdot 4+\ldots+(k+1)(k+2)$

$=\frac{1}{3}(k+1)(k+2)(k+3).$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2) \text{ for all } n\in N.$$

EXAMPLE 6 *Using the principle of mathematical induction, prove that*

$$1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1).$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1).$$

When $n=1$, we have

LHS $=1\cdot 3=3$ and RHS $=\frac{1}{3}\times 1\times(4\times 1^2+6\times 1-1)=\frac{1}{3}\times 1\times 9=3.$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ 1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2k-1)(2k+1)=\frac{1}{3}k(4k^2+6k-1). \quad \ldots \text{(i)}$$

Now, $1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2k-1)(2k+1)+\{2(k+1)-1\}\{2(k+1)+1\}$

$$=\{1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2k-1)(2k+1)\}+(2k+1)(2k+3)$$

$$=\frac{1}{3}k(4k^2+6k-1)+(2k+1)(2k+3) \qquad \text{[using (i)]}$$

$$=\frac{1}{3}[(4k^3+6k^2-k)+3(4k^2+8k+3)]=\frac{1}{3}(4k^3+18k^2+23k+9)$$

$$=\frac{1}{3}(k+1)(4k^2+14k+9)=\frac{1}{3}(k+1)[4(k+1)^2+6(k+1)-1].$$

$\therefore$ $P(k+1):\ 1\cdot 3+3\cdot 5+5\cdot 7+\ldots+\{2(k+1)-1\}\{2(k+1)+1\}$

$$=\frac{1}{3}(k+1)\{4(k+1)^2+6(k+1)-1\}.$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1)$ for all $n\in N$.

EXAMPLE 7 *Using the principle of mathematical induction, prove that*

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$$

for all $n\in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3).$$

When $n=1$, we have

$$\text{LHS}=1\cdot 2\cdot 3=6 \quad \text{and} \quad \text{RHS}=\frac{1}{4}\times 1\times 2\times 3\times 4=6.$$

$\therefore$ LHS = RHS.

Thus, the given statement is true for $n=1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+k(k+1)(k+2)$$

$$=\frac{1}{4}k(k+1)(k+2)(k+3). \qquad \ldots \text{(i)}$$

Now, $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+k(k+1)(k+2)+(k+1)(k+2)(k+3)$

$$=\{1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+k(k+1)(k+2)\}+(k+1)(k+2)(k+3)$$

$$=\frac{1}{4}k(k+1)(k+2)(k+3)+\frac{4(k+1)(k+2)(k+3)}{4} \qquad \text{[using (i)]}$$

$$=\frac{1}{4}(k+1)(k+2)(k+3)(k+4).$$

$\therefore \quad P(k+1)\colon\ 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+(k+1)(k+2)(k+3)$

$$=\frac{1}{4}(k+1)(k+2)(k+3)(k+4).$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

P(1) is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$$

for all values of $n\in N$.

EXAMPLE 8 *Using the principle of mathematical induction, prove that*

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{(n+1)}.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n)\colon\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{(n+1)}.$$

Putting $n=1$ in the given statement, we get

$$\text{LHS}=\frac{1}{1\cdot 2}=\frac{1}{2}\ \text{ and }\ \text{RHS}=\frac{1}{(1+1)}=\frac{1}{2}.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k)\colon\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{(k+1)}. \qquad \ldots\text{(i)}$$

Now, $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$

$$=\left\{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}\right\}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)} \qquad \text{[using (i)]}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{(k+1)}{(k+2)}.$$

$$\therefore\quad P(k+1)\colon\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{1}{(k+1)(k+2)}=\frac{(k+1)}{(k+2)}.$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ *P*(1) is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}\ \text{for all}\ n\in N.$$

EXAMPLE 9 *Using the principle of mathematical induction, prove that*

$$\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ \frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS}=\frac{1}{3\cdot 5}=\frac{1}{15} \text{ and } \text{RHS}=\frac{1}{3(2\times 1+3)}=\frac{1}{15}.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ \frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)}. \ldots \text{(i)}$$

Now, $\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$

$$=\left\{\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k+1)(2k+3)}\right\}+\frac{1}{(2k+3)(2k+5)}$$

$$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \qquad \text{[using (i)]}$$

$$=\frac{k(2k+5)+3}{3(2k+3)(2k+5)}=\frac{(2k^2+5k+3)}{3(2k+3)(2k+5)}=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)}{3(2k+5)}=\frac{(k+1)}{3\{2(k+1)+3\}}.$$

$\therefore$ $P(k+1):\ \frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$

$$=\frac{(k+1)}{3\{2(k+1)+3\}}.$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

for all values of $n\in N$.

EXAMPLE 10 *Using the principle of mathematical induction, prove that*

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)} \text{ for all } n\in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS}=\frac{1}{1\cdot 2\cdot 3}=\frac{1}{6} \text{ and } \text{RHS}=\frac{1\times(1+3)}{4\times(1+1)(1+2)}=\frac{1\times 4}{4\times 2\times 3}=\frac{1}{6}.$$

$\therefore$ LHS = RHS.

Thus, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}. \quad \ldots \text{(i)}$$

Now, $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$$=\left\{\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}\right\}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\left\{\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}\right\} \qquad \text{[using (i)]}$$

$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}=\frac{(k^3+6k^2+9k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}.$$

$\therefore$ $P(k+1):\ \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{(k+1)(k+2)(k+3)}$

$$=\frac{(k+1)(k+4)}{4(k+2)(k+3)}.$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

for all values of $n \in N$.

EXAMPLE 11 *Using the principle of mathematical induction, prove that*

$$1\cdot 3+2\cdot 3^2+3\cdot 3^3+\ldots+n\cdot 3^n=\frac{(2n-1)3^{n+1}+3}{4} \text{ for all } n\in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1\cdot 3+2\cdot 3^2+3\cdot 3^3+\ldots+n\cdot 3^n=\frac{(2n-1)3^{n+1}+3}{4}.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS}=1\cdot 3=3 \text{ and } \text{RHS}=\frac{(2\times 1-1)3^{1+1}+3}{4}=\frac{1\times 9+3}{4}=\frac{12}{4}=3.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ 1\cdot 3+2\cdot 3^2+3\cdot 3^3+\ldots+k\cdot 3^k=\frac{(2k-1)3^{k+1}+3}{4}. \quad \ldots \text{(i)}$$

$\therefore$ $1\cdot 3+2\cdot 3^2+3\cdot 3^3+\ldots+k\cdot 3^k+(k+1)3^{(k+1)}$

$$= \{1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + k\cdot 3^k\} + (k+1)3^{(k+1)}$$

$$= \frac{(2k-1)3^{k+1}+3}{4} + (k+1)3^{k+1} \qquad \text{[using (i)]}$$

$$= \frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4} = \frac{(2k-1+4k+4)3^{k+1}+3}{4}$$

$$= \frac{(6k+3)3^{k+1}+3}{4} = \frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}.$$

$\therefore$ $P(k+1)$: $1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + (k+1)\cdot 3^{k+1}$

$$= \frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}.$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + n\cdot 3^n = \frac{(2n-1)3^{n+1}+3}{4}$$

for all values of $n \in N$.

EXAMPLE 12 *Using the principle of mathematical induction, prove that*

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n+1}\right) = \frac{1}{(n+1)} \text{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n): \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n+1}\right) = \frac{1}{(n+1)}.$$

When $n = 1$, LHS $= \left(1-\frac{1}{2}\right) = \frac{1}{2}$ and RHS $= \frac{1}{(1+1)} = \frac{1}{2}$.

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{k+1}\right) = \frac{1}{(k+1)}. \qquad \ldots \text{(i)}$$

Now, $\left\{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{k+1}\right)\right\}\cdot\left(1-\frac{1}{k+2}\right)$

$$= \frac{1}{(k+1)}\cdot\left[\frac{(k+2)-1}{(k+2)}\right] = \frac{1}{(k+1)}\cdot\frac{(k+1)}{(k+2)} = \frac{1}{(k+2)} \qquad \text{[using (i)].}$$

$\therefore$ $P(k+1)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{k+2}\right) = \frac{1}{(k+2)}.$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n+1}\right) = \frac{1}{(n+1)} \text{ for all } n \in N.$$

EXAMPLE 13 *Using the principle of mathematical induction, prove that*

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{(r-1)} \text{ for } r > 1 \text{ and all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{(r-1)}.$$

When $n = 1$, we have

$$\text{LHS} = a \text{ and } \text{RHS} = \frac{a(r^1 - 1)}{(r-1)} = a.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(r^k - 1)}{(r-1)}. \quad \ldots \text{(i)}$$

Now, $(a + ar + ar^2 + \ldots + ar^{k-1}) + ar^k = \frac{a(r^k - 1)}{(r-1)} + ar^k$ [using (i)]

$$= \frac{a(r^{k+1} - 1)}{(r-1)}.$$

$\therefore$ $P(k+1):\ a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = \frac{a(r^{k+1} - 1)}{(r-1)}.$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{(r-1)} \text{ for } r > 1 \text{ and all } n \in N.$$

EXAMPLE 14 *Let a and b be arbitrary real numbers. Using the principle of mathematical induction, prove that*

$$(ab)^n = a^n b^n \text{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ (ab)^n = a^n b^n.$$

When $n = 1$, we have

$$\text{LHS} = (ab)^1 = ab \text{ and } \text{RHS} = a^1 b^1 = ab.$$

$\therefore$ LHS = RHS.

Thus, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ (ab)^k = a^k b^k. \quad \ldots \text{(i)}$$

Now, $(ab)^{k+1} = (ab)^k (ab) = (a^k b^k)(ab)$ [using (i)]

$= (a^k \cdot a)(b^k \cdot b)$ [by commutativity and associativity of multiplication on real numbers]

$= (a^{k+1} \cdot b^{k+1})$.

$\therefore$ $P(k+1)$: $(ab)^{k+1} = (a^{k+1} \cdot b^{k+1})$.

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$(ab)^n = a^n b^n$ for all $x \in N$.

EXAMPLE 15 *Using the principle of mathematical induction, prove that* (n^2+n) *is even for all* $n \in N$.

SOLUTION Let $P(n)$: (n^2+n) is even.

For $n = 1$, the given expression becomes $(1^2+1) = 2$, which is even.

So, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: (k^2+k) is even

$\Rightarrow$ $(k^2+k) = 2m$ for some natural number *m*. ... (i)

Now, $(k+1)^2+(k+1) = k^2+3k+2 = (k^2+k)+2(k+1)$

$= 2m+2(k+1)$ [using (i)]

$= 2[m+(k+1)]$, which is clearly even.

$\therefore$ $P(k+1)$: $(k+1)^2+(k+1)$ is even.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of induction, it follows that (n^2+n) is even for all $n \in N$.

EXAMPLE 16 *Using the principle of mathematical induction, prove that* $n(n+1)(n+5)$ *is a multiple of* 3 *for all* $n \in N$.

SOLUTION Let $P(n)$: $n(n+1)(n+5)$ is a multiple of 3.

For $n = 1$, the given expression becomes $(1 \times 2 \times 6) = 12$, which is a multiple of 3.

So, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $k(k+1)(k+5)$ is a multiple of 3

$\Rightarrow$ $k(k+1)(k+5) = 3m$ for some natural number *m*. ... (i)

Now, $(k+1)(k+2)(k+6)$

$= (k+1)(k+2)k+6(k+1)(k+2)$

$= k(k+1)(k+5-3)+6(k+1)(k+2)$

$= k(k+1)(k+5)-3k(k+1)+6(k+1)(k+2)$

$= 3m+3(k+1)(2k+4-k)$

$= 3m+3(k+1)(k+4)$

$= 3\{m+(k+1)(k+4)\}$, which is a multiple of 3.

$\therefore$ $P(k+1)$: $(k+1)(k+2)(k+6)$ is a multiple of 3.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $n(n+1)(n+5)$ is a multiple of 3 for all $n \in N$.

EXAMPLE 17 *Using the principle of mathematical induction, prove that* $(7^n - 3^n)$ *is divisible by* 4 *for all* $n \in N$.

SOLUTION Let $P(n)$: $(7^n - 3^n)$ is divisible by 4.

For $n = 1$, the given expression becomes $(7^1 - 3^1) = 4$, which is divisible by 4.

So, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(7^k - 3^k)$ is divisible by 4.

$\Rightarrow$ $(7^k - 3^k) = 4m$ for some natural number m. ... (i)

Now, $\{7^{(k+1)} - 3^{(k+1)}\}$

$= 7^{(k+1)} - 7 \cdot 3^k + 7 \cdot 3^k - 3^{(k+1)}$ [subtracting and adding $7 \cdot 3^k$]

$= 7(7^k - 3^k) + 3^k(7 - 3)$

$= (7 \times 4m) + 4 \cdot 3^k$ [using (i)]

$= 4(7m + 3^k)$, which is clearly divisible by 4.

$\therefore$ $P(k+1)$: $\{7^{(k+1)} - 3^{(k+1)}\}$ is divisible by 4.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $(7^n - 3^n)$ is divisible by 4 for all values of $n \in N$.

EXAMPLE 18 *Using the principle of mathematical induction, prove that* $(10^{2n-1} + 1)$ *is divisible by* 11 *for all* $n \in N$.

SOLUTION Let $P(n)$: $(10^{2n-1} + 1)$ is divisible by 11.

For $n = 1$, the given expression becomes $\{10^{(2 \times 1 - 1)} + 1\} = 11$, which is divisible by 11.

So, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(10^{2k-1} + 1)$ is divisible by 11

$\Rightarrow$ $(10^{2k-1} + 1) = 11m$, for some natural number m. ... (i)

Now, $\{10^{2(k+1)-1} + 1\} = \{10^2 \cdot 10^{(2k-1)} + 1\}$

$= 100 \times \{10^{2k-1} + 1\} - 99$

$= (100 \times 11m) - 99$ [using (i)]

$= 11 \times (100m - 9)$, which is divisible by 11.

$\therefore$ $P(k+1)$: $\{10^{2(k+1)-1} + 1\}$ is divisible by 11.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $\{10^{2n-1}+1\}$ is divisble by 11 for all $n \in N$.

EXAMPLE 19 *Using the principle of mathematical induction, prove that* $(2\cdot 7^n + 3\cdot 5^n - 5)$ *is divisible by* 24 *for all* $n \in N$.

SOLUTION Let $P(n)$: $(2\cdot 7^n + 3\cdot 5^n - 5)$ is divisible by 24.

For $n=1$, the given expression becomes $(2\cdot 7^1 + 3\cdot 5^1 - 5) = 24$, which is clearly divisible by 24.

So, the given statement is true for $n=1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(2\cdot 7^k + 3\cdot 5^k - 5)$ is divisible by 24

$\Rightarrow$ $(2\cdot 7^k + 3\cdot 5^k - 5) = 24m$, for some $m \in N$. ... (i)

Now, $(2\cdot 7^{k+1} + 3\cdot 5^{k+1} - 5)$

$= (2\cdot 7^k \cdot 7 + 3\cdot 5^k \cdot 5 - 5)$

$= 7(2\cdot 7^k + 3\cdot 5^k - 5) - 6\cdot 5^k + 30$

$= (7\times 24m) - 6(5^k - 5)$ [using (i)]

$= (24\times 7m) - 6\times 5\times (5^{k-1} - 1)$

$= (24\times 7m) - 5\times 24p$ $\left[\because\ (5^{k-1}-1) \text{ is divisible by } (5-1), \text{ i.e., } 4 \Rightarrow (5^{k-1}-1) = 4p\right]$

$= 24\times(7m-5p)$, which is clearly divisible by 24.

$\therefore$ $P(k+1)$: $(2\cdot 7^{k+1} + 3\cdot 5^{k+1} - 5)$ is divisible by 24.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $(2\cdot 7^n + 3\cdot 5^n - 5)$ is divisible by 24 for all $n \in N$.

EXAMPLE 20 *Using the principle of mathematical induction, prove that* $(x^n - y^n)$ *is divisible by* $(x-y)$ *for all* $n \in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n)$: $(x^n - y^n)$ is divisible by $(x-y)$.

When $n=1$, the given statement becomes: $(x^1 - y^1)$ is divisible by $(x-y)$, which is clearly true.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(x^k - y^k)$ is divisible by $(x-y)$. ... (i)

Now, $(x^{k+1} - y^{k+1})$

$= \{x^{k+1} - x^k y + x^k y - y^{k+1}\}$ [on adding and subtracting $x^k y$]

$= x^k(x-y) + y(x^k - y^k)$, which is divisible by $(x-y)$ [using (i)].

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $(x^n - y^n)$ is divisible by $(x-y)$ for all $n \in N$.

EXAMPLE 21 *Using the principle of mathematical induction, prove that* $(1+x)^n \geq (1+nx)$ *for all* $n \in N$, *where* $x > -1$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n)$: $(1+x)^n \geq (1+nx)$, where $x > -1$.

When $n = 1$, we have

LHS $= (1+x)^1 = (1+x)$ and RHS $= (1+1\times x) = (1+x)$.

$\therefore$ LHS $\geq$ RHS is true.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(1+x)^k \geq (1+kx)$, where $x > -1$. ... (i)

Now, $(1+x)^{k+1} = (1+x)^k(1+x)$

$\geq (1+kx)(1+x)$ [using (i)]

$= 1 + (k+1)x + kx^2$

$\geq 1 + (k+1)x$ [$\because$ $kx^2 \geq 0$].

$\therefore$ $P(k+1)$: $(1+x)^{k+1} \geq 1 + (k+1)x$.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$(1+x)^n \geq (1+nx)$, where $x > -1$ for all $n \in N$.

EXAMPLE 22 *Using the principle of mathematical induction, prove that* $n < 2^n$ *for all* $n \in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n)$: $n < 2^n$.

When $n = 1$, we get LHS $= 1$ and RHS $= 2^1 = 2$.

Clearly, $1 < 2$.

$\therefore$ $P(n)$ is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $k < 2^k$.

Now, $k < 2^k \Rightarrow 2k < 2^{k+1}$

$\Rightarrow (k+k) < 2^{k+1}$

$\Rightarrow (k+1) \leq (k+k) < 2^{k+1}$ [$\because$ $1 \leq k$]

$\Rightarrow (k+1) < 2^{k+1}$.

$\therefore$ $P(k+1)$: $(k+1) < 2^{k+1}$.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $n < 2^n$ for all $n \in N$.

EXAMPLE 23 *Using the principle of mathematical induction, prove that*

$(2n+7) < (n+3)^2$ *for all values of* $n \in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n)$: $(2n+7) < (n+3)^2$.

When $n = 1$, we have

LHS $= (2 \times 1 + 7) = 9$ and RHS $= (1+3)^2 = 4^2 = 16$.

$\therefore$ LHS < RHS $[\because\ 9 < 16]$.

Thus, $P(n)$ is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(2k+7) < (k+3)^2$. ... (i)

Now, $2(k+1)+7 = (2k+7)+2$

$< (k+3)^2 + 2 = (k^2 + 6k + 11)$ [using (i)]

$< (k^2 + 8k + 16) = (k+4)^2 = (k+1+3)^2$.

$\therefore$ $P(k+1)$: $2(k+1)+7 < (k+1+3)^2$.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $(2n+7) < (n+3)^2$ for all $n \in N$.

EXAMPLE 24 *Using the principle of mathematical induction, prove that*

$(1^2 + 2^2 + \ldots + n^2) > \dfrac{n^3}{3}$ *for all values of* $n \in N$.

SOLUTION Let $P(n)$: $(1^2 + 2^2 + \ldots + n^2) > \dfrac{n^3}{3}$.

When $n = 1$, LHS $= 1^2 = 1$ and RHS $= \dfrac{1^3}{3} = \dfrac{1}{3}$.

Since $1 > \dfrac{1}{3}$, it follows that $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(1^2 + 2^2 + \ldots + k^2) > \dfrac{k^3}{3}$. ... (i)

Now, $1^2 + 2^2 + \ldots + k^2 + (k+1)^2$

$= \{1^2 + 2^2 + \ldots + k^2\} + (k+1)^2$

$> \dfrac{k^3}{3} + (k+1)^2$ [using (i)]

$= \dfrac{1}{3} \cdot \{k^3 + 3(k+1)^2\} = \dfrac{1}{3} \cdot \{k^3 + 3k^2 + 6k + 3\}$

$$= \frac{1}{3}[\{(k^3 + 1 + 3k(k+1)\} + (3k+2)] = \frac{1}{3} \cdot [(k+1)^3 + (3k+2)]$$

$$> \frac{1}{3}(k+1)^3.$$

$\therefore \quad P(k+1)\colon\ 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 > \frac{1}{3}(k+1)^3.$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that

$(1^2 + 2^2 + 3^2 + \ldots + n^2) > \frac{n^3}{3}$ for all $n \in N$.

EXAMPLE 25 *Using the principle of mathematical induction, prove that*

$(1 + 2 + 3 + \ldots + n) < \frac{1}{8}(2n+1)^2$ *for all values of* $n \in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n)\colon\ (1 + 2 + 3 + \ldots + n) < \frac{1}{8}(2n+1)^2.$

When $n = 1$, we have

LHS = 1 and RHS $= \frac{1}{8}(2 \times 1 + 1)^2 = \frac{9}{8}$.

Clearly, $1 < \frac{9}{8}$.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)\colon\ (1 + 2 + 3 + \ldots + k) < \frac{1}{8}(2k+1)^2.$ ... (i)

Now, $1 + 2 + 3 + \ldots + k + (k+1)$

$= \{1 + 2 + 3 + \ldots + k\} + (k+1)$

$< \frac{1}{8}(2k+1)^2 + (k+1) = \frac{(2k+1)^2 + 8(k+1)}{8}$ [using (i)]

$= \frac{(4k^2 + 12k + 9)}{8} = \frac{(2k+3)^2}{8} = \frac{\{2(k+1)+1\}^2}{8}$.

$\therefore \quad \{1 + 2 + 3 + \ldots + k + (k+1)\} < \frac{\{2(k+1)+1\}^2}{8}$.

This shows that $P(k+1)$ is true.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that

$(1 + 2 + 3 + \ldots + n) < \frac{1}{8}(2n+1)^2$ for all values of $n \in N$.

EXERCISE 4

Using the principle of mathematical induction, prove each of the following for all $n \in N$:

1. $1+2+3+4+\ldots+n=\frac{1}{2}n(n+1)$.

2. $2+4+6+8+\ldots+2n=n(n+1)$.

3. $1+3+3^2+3^3+\ldots+3^{n-1}=\frac{1}{2}(3^n-1)$.

4. $2+6+18+\ldots+2\cdot 3^{n-1}=(3^n-1)$.

5. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=\left(1-\frac{1}{2^n}\right)$.

6. $1^2+3^2+5^2+7^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$.

7. $1\cdot 2+2\cdot 2^2+3\cdot 2^3+\ldots+n\cdot 2^n=(n-1)2^{n+1}+2$.

8. $3\cdot 2^2+3^2\cdot 2^3+3^3\cdot 2^4+\ldots+3^n\cdot 2^{n+1}=\frac{12}{5}(6^n-1)$.

9. $1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots+n)}=\frac{2n}{(n+1)}$.

10. $\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$.

11. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$.

12. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{(2n+1)}$.

13. $\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$.

14. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left\{1+\frac{(2n+1)}{n^2}\right\}=(n+1)^2$.

15. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$.

16. $n(n+1)(n+2)$ is a multiple of 6.

17. $(x^{2n}-y^{2n})$ is divisible by $(x+y)$.

18. $(x^{2n}-1)$ is divisible by $(x-1)$, where $x \neq 1$.

19. $\{(41)^n-(14)^n\}$ is divisible by 27.

20. $(4^n+15n-1)$ is divisible by 9.

21. $(3^{2n+2}-8n-9)$ is divisible by 8.

22. $(2^{3n}-1)$ is a multiple of 7.

23. $3^n \geq 2^n$

6. $P(k)$: $1^2+3^2+5^2+\ldots+(2k-1)^2=\dfrac{k(2k-1)(2k+1)}{3}$.

Now, $\{1^2+3^2+5^2+\ldots+(2k-1)^2\}+\{2(k+1)-1\}^2$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2=\frac{1}{3}\cdot\{k(2k-1)(2k+1)+3(2k+1)^2\}$$

$$=\frac{1}{3}(2k+1)\{k(2k-1)+3(2k+1)\}=\frac{1}{3}(2k+1)(2k^2+5k+3)$$

$$=\frac{1}{3}(k+1)(2k+1)(2k+3).$$

7. $P(k)$: $1\cdot2+2\cdot2^2+3\cdot2^3+\ldots+k\cdot2^k=(k-1)\cdot2^{k+1}+2.$ … (i)

Now, $\{1\cdot2+2\cdot2^2+3\cdot2^3+\ldots+k\cdot2^k\}+(k+1)\cdot2^{k+1}$

$$=\{(k-1)\cdot2^{k+1}+2\}+(k+1)\cdot2^{k+1}=2k\cdot2^{k+1}+2=k\cdot2^{k+2}+2.$$

8. $P(k)$: $3\cdot2^2+3^2\cdot2^3+\ldots+3^k\cdot2^{k+1}=\dfrac{12}{5}(6^k-1)$.

Now, $(3\cdot2^2+3^2\cdot2^3+\ldots+3^k\cdot2^{k+1})+3^{k+1}\cdot2^{k+2}$

$$=\left\{\frac{12}{5}(6^k-1)+3^{k+1}\cdot2^{k+1}\cdot2\right\}=\frac{1}{5}\cdot\{12(6^k-1)+10\times6^{k+1}\}$$

$$=\frac{1}{5}\cdot\{2(6^{k+1}-6)+10\times6^{k+1}\}=\frac{12}{5}(6^{k+1}-1).$$

14. $P(k)$: $(1+3)\left(1+\dfrac{5}{4}\right)\left(1+\dfrac{7}{9}\right)\ldots\left\{1+\dfrac{(2k+1)}{k^2}\right\}=(k+1)^2$.

Now, $(1+3)\left(1+\dfrac{5}{4}\right)\ldots\left(1+\dfrac{2k+1}{k^2}\right)\left\{1+\dfrac{2(k+1)+1}{(k+1)^2}\right\}$

$$=(k+1)^2\times\frac{(k+1)^2+(2k+3)}{(k+1)^2}=(k^2+4k+4)=(k+2)^2.$$

15. $P(k)$: $(1+1)\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)\ldots\left(1+\dfrac{1}{k}\right)=(k+1)$.

Now, $\left\{(1+1)\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)\ldots\left(1+\dfrac{1}{k}\right)\right\}\left(1+\dfrac{1}{k+1}\right)$

$$=(k+1)\left(1+\frac{1}{k+1}\right)=(k+1)\cdot\frac{(k+2)}{(k+1)}=(k+2).$$

16. Let $k(k+1)(k+2)=6p$ … (i). Then,

$(k+1)(k+2)(k+3)=(k+1)(k+2)k+3(k+1)(k+2)=k(k+1)(k+2)+3(k+1)(k+2)$

$=6p+3(k+1)(k+2)$ [using (i)]

$=6p+6q=6(p+q)$, where $(k+1)(k+2)=2q$

{$\because$ $(k+1)$ and $(k+2)$ being consecutive integers, one of them must be even and so their product is even}.

17. Let $\dfrac{(x^{2n}-y^{2n})}{(x+y)}=p$ … (i). Then,

$\{x^{2(n+1)}-y^{2(n+1)}\}=(x^{2n}\cdot x^2-y^{2n}\cdot y^2)=x^{2n}\cdot x^2-x^2y^{2n}+x^2y^{2n}-y^{2n}\cdot y^2$

$= x^2(x^{2n} - y^{2n}) + y^{2n}(x^2 - y^2) = x^2 p(x+y) + y^{2n}(x^2 - y^2)$ [using (i)]

$= (x+y)[x^2 p + y^{2n}]$, which is divisible by $(x+y)$.

18. Let $\frac{(x^{2n}-1)}{(x-1)} = p$. … (i). Then,

$\{x^{2(n+1)} - 1\} = (x^{2n} \cdot x^2 - x^2 + x^2 - 1) = x^2(x^{2n} - 1) + (x^2 - 1)$

$= x^2 \cdot p(x-1) + (x^2 - 1)$ [using (i)]

$= (x-1)(px^2 + x + 1)$, which is divisible by $(x-1)$.

19. Let $\frac{(41)^k - (14)^k}{27} = p$. … (i). Then,

$\{(41)^{k+1} - (14)^{k+1}\} = (41)^{k+1} - (41)^k \cdot 14 + (41)^k \cdot 14 - (14)^{k+1}$

$= (41)^k(41 - 14) + 14\{(41)^k - (14)^k\} = 27 \times (41)^k + 14 \times 27p$

$= 27 \times [(41)^k + 14]$.

20. Let $(4^k + 15k - 1) = 9p$ … (i). Then,

$\{4^{(k+1)} + 15(k+1) - 1\} = \{4^{(k+1)} + 15k + 14\} = (4 \cdot 4^k + 60k - 4) - 45k + 18$

$= 4(4^k + 15k - 1) - 9(5k - 2) = (4 \times 9p) - 9(5k - 2)$ [using (i)]

$= 9 \times (4p - 5k + 2)$, which is divisible by 9.

21. Let $(3^{2k+2} - 8k - 9) = 8p$ … (i). Then,

$\{3^{2(k+1)+2} - 8(k+1) - 9\} = \{3^{(2k+2)} \cdot 3^2 - 8k - 17\} = (9 \cdot 3^{(2k+2)} - 72k - 81) + (64k + 64)$

$= 9 \cdot \{3^{(2k+2)} - 8k - 9\} + 64(k+1)$

$= (9 \times 8p) + 64(k+1)$ [using (i)]

$= 8 \times [9p + 8(k+1)]$, which is divisible by 8.

22. Let $(2^{3k} - 1) = 7p$ … (i). Then,

$\{2^{3(k+1)} - 1\} = (2^{3k} \cdot 2^3 - 1) = (8 \cdot 2^{3k} - 8) + 7 = \{8(2^{3k} - 1) + 7\} = (8 \times 7p) + 7$ [using (i)]

$= (56p + 7) = 7(8p + 1)$, which is divisible by 7.

23. Clearly, $3^1 \geq 2^1$. So, the result is true for $n = 1$.

Let it be true for $n = k$. Then,

$3^k \geq 2^k$ and $3 > 2 \Rightarrow 3^k \cdot 3 \geq 2^k \cdot 2 \Rightarrow 3^{k+1} \geq 2^{k+1}$.

So, whenever the result is true for *k*, then it is also true for $(k+1)$.

SUMMARY OF KEY FACTS

PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(i) *P*(1) *is true and (ii) $P(k+1)$ is true, whenever $P(k)$ is true then $P(n)$ is true for all $n \in N$.*